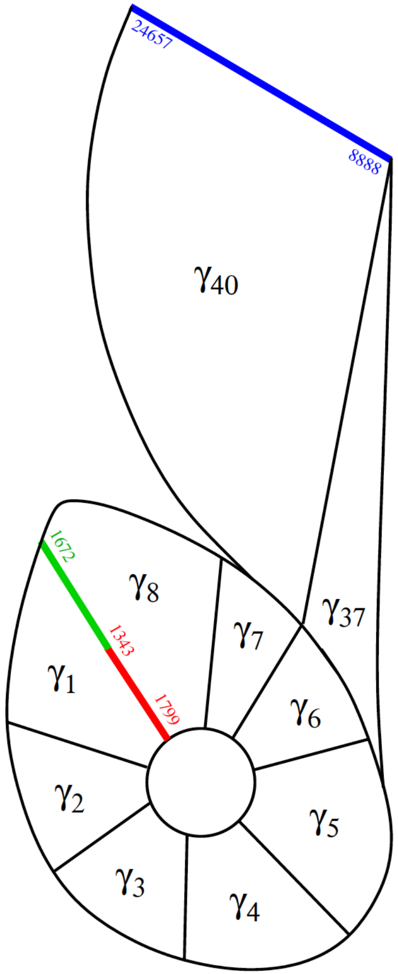
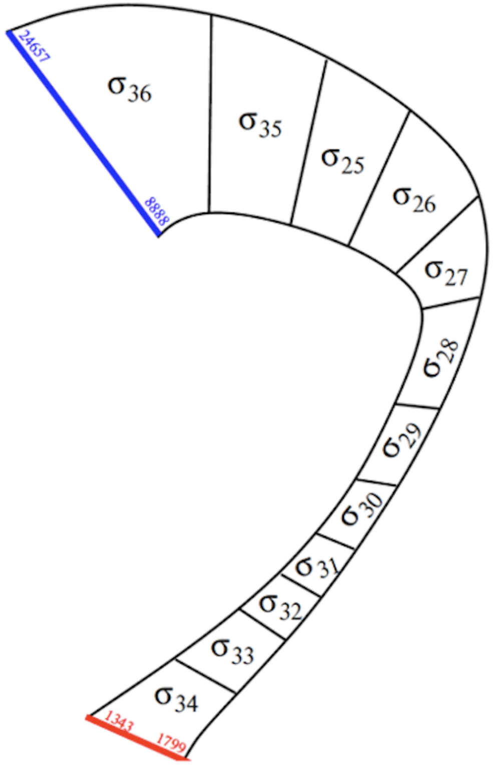
Università di Pisa

Metodi matematici per la scienza del clima



Minicourse “Dynamical Systems, Algebraic Topology, and Climate”

by Michael Ghil and
Denisse Sciamarella



Topology as a theoretical and data analysis tool for understanding the fundamental processes underlying the dynamics of complex systems.

This review article is based on the invited talks given by the two authors in an online series on "Perspectives on climate sciences: From historical developments to research frontiers".

Nonlin. Processes Geophys., 30, 399–434, 2023
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Nonlinear Processes in Geophysics
Open Access

Review article

Review article: Dynamical systems, algebraic topology and the climate sciences

Michael Ghil[1,2,3] **and Denisse Sciamarella**[4,5,6]

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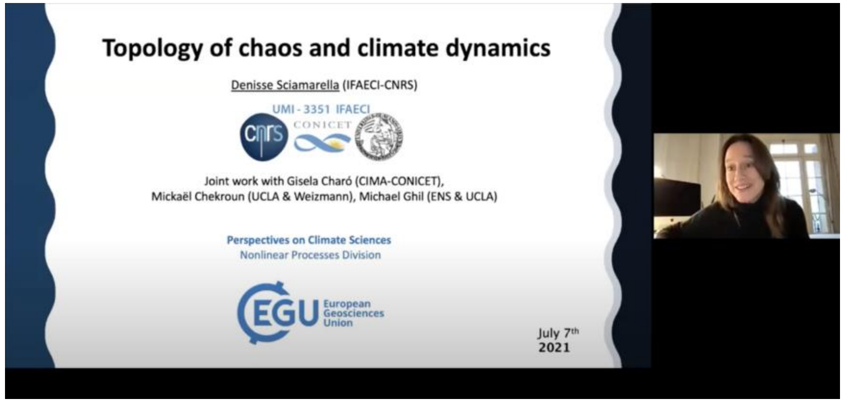


Dynamical Systems, Algebraic Topology, and Climate

Introduction

- What is phase space topology?

Methods

- Branched manifolds, cell complexes, homologies.

Applications

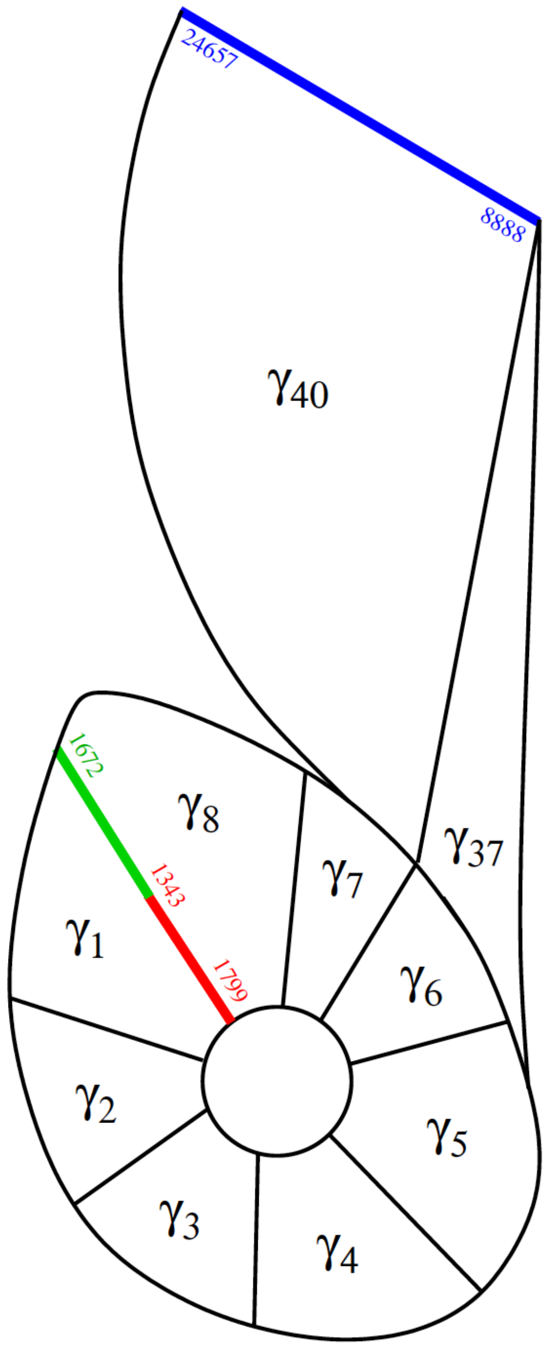
- Lagrangian analysis, Climate dynamics.

Templex

- Why and how was it conceived? How is it computed?

Random templex

- How is it defined? How does it encode topological tipping points?



Introduction

- What is phase space topology and why is it important?

The first sentence of Leo Tolstoy's novel Anna Karenina is:

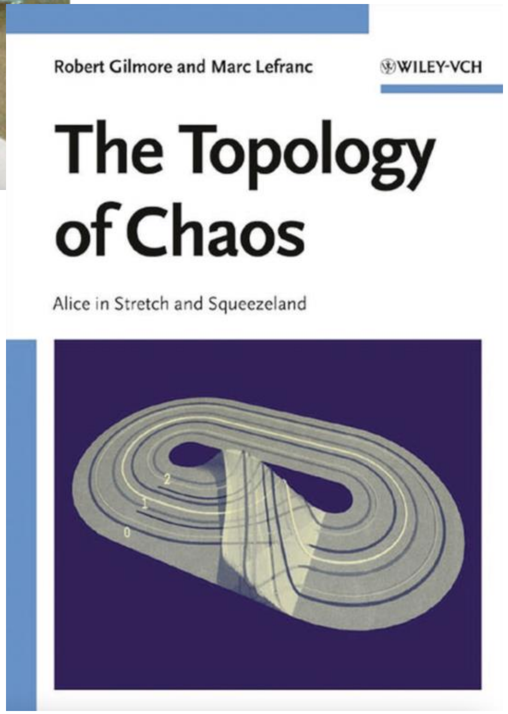
"All happy families are alike; each unhappy family is unhappy in its own way".

Following the famous writer, Robert Gilmore and Marc Lefranc, tell us:

"All linear systems are alike; each nonlinear system is nonlinear in its own way".

“It was a very happy and shocking discovery that there were structures in nonlinear systems that are always the same if you looked at them the right way.”

This talk will show that topology is <u>the right way</u> of looking at dynamical systems.



Metodi matematici per la scienza del clima

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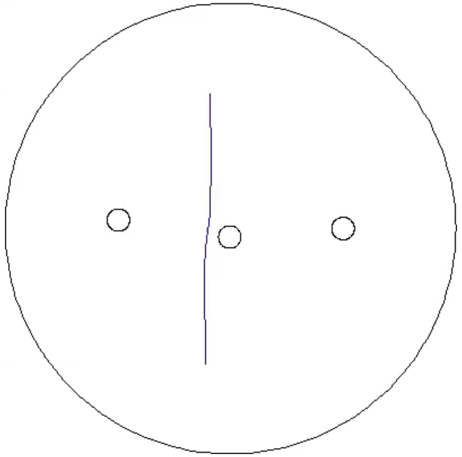
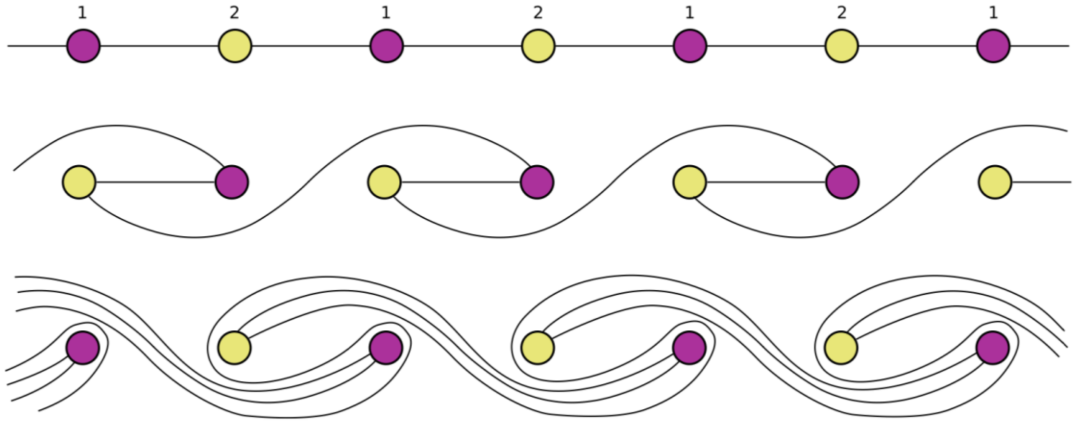
Introduction

- Phase space topology is not topological chaos

Topological chaos considers the fluid mechanics problem of how fluid particle trajectories are entangled in physical space during a mixing experiment.





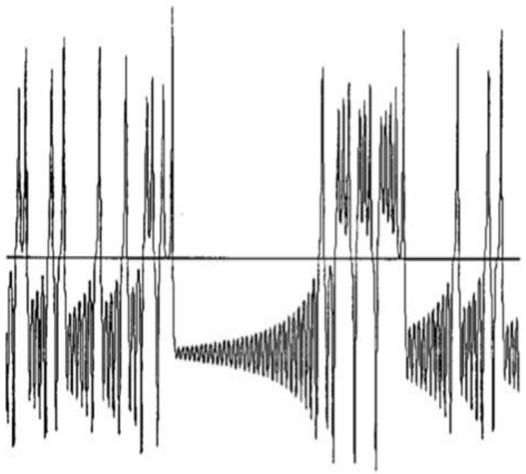
Topological mixing with ghost rods
Gouillart *et al*. PRE 73, 036311 (2006)

It generally relies on the periodic motion of obstacles in a two-dimensional flow in order to form nontrivial braids. This motion generates exponential stretching of material lines, and hence efficient mixing.

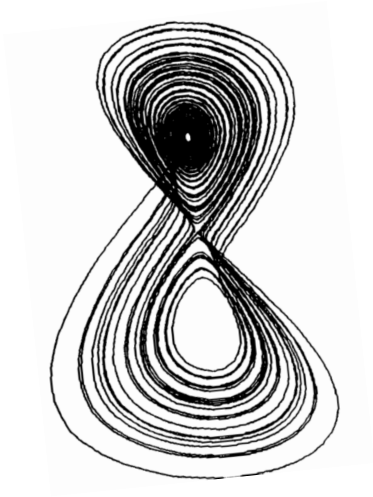
Introduction

- What is phase space topology and why is it important?

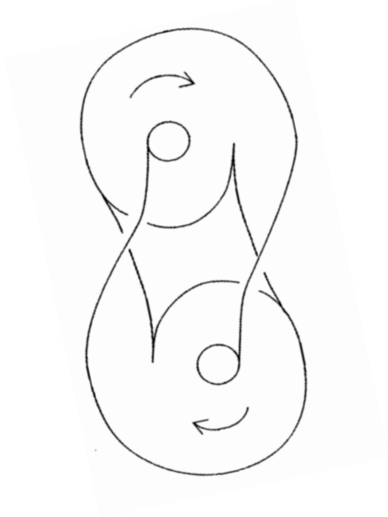
Phase space topology (also, chaos topology or topology of chaos) considers how *n*-dimensional trajectories and point clouds representing a flow are structured in phase space within Dynamical Systems Theory.



Optically pumped molecular laser run under a resonance-operating condition.



Embedding projection onto a plane.



Branched manifold.

R. Gilmore, Reviews of Modern Physics, Vol. 70, No. 4, October 1998

Introduction

- What is phase space topology and why is it important?

The first methods to reconstruct dynamic configurations in phase space from experimental time series and to study geometric structures in this space appear in 1980.

VOLUME 45, NUMBER 9 PHYSICAL REVIEW LETTERS 1 SEPTEMBER 1980

Geometry from a Time Series

N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw
Dynamical Systems Collective, Physics Department, University of California, Santa Cruz, California 95064
(Received 13 November 1979)

It is shown how the existence of low-dimensional chaotic dynamical systems describing turbulent fluid flow might be determined experimentally. Techniques are outlined for reconstructing phase-space pictures from the observation of a single coordinate of any dissipative dynamical system, and for determining the dimensionality of the system's attractor. These techniques are applied to a well-known simple three-dimensional chaotic dynamical system.

PACS numbers: 47.25.-c

Introduction

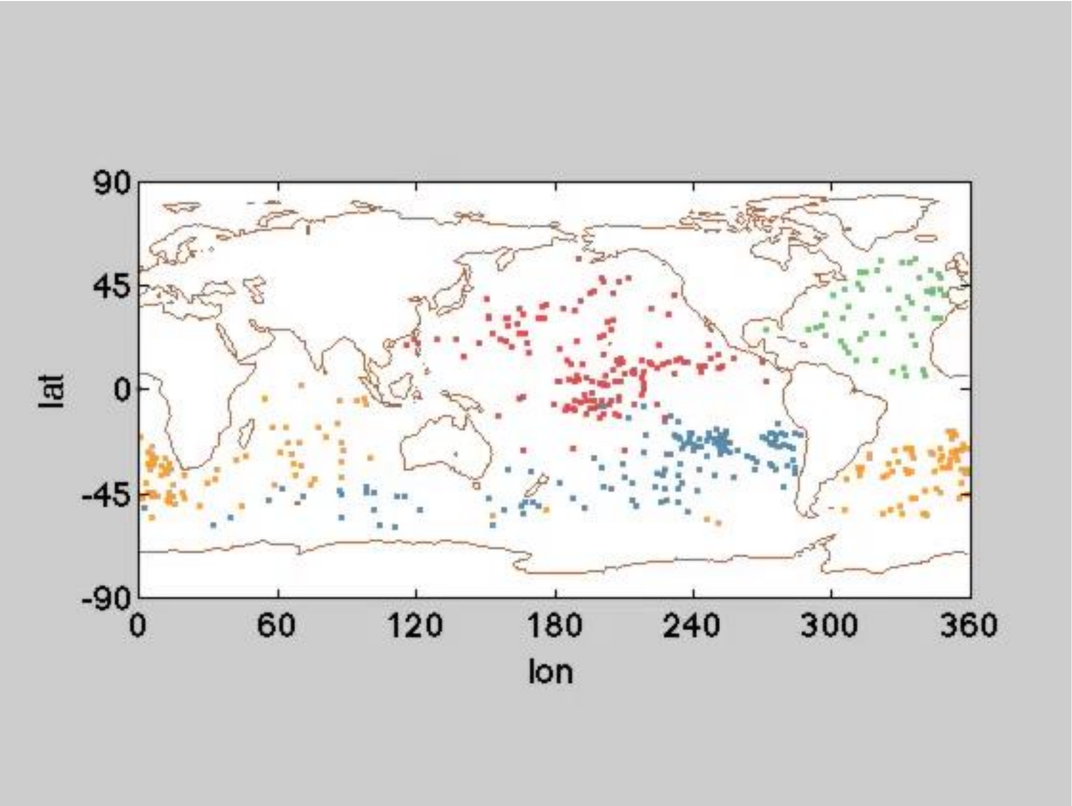
- What is phase space topology and why is it important?

Geometric methods continue to be used, e.g., to understand datasets of Lagrangian trajectories.

Understanding the geometry of transport: Diffusion maps for Lagrangian trajectory data unravel coherent sets

Cite as: Chaos **27**, 035804 (2017); https://doi.org/10.1063/1.4971788
Submitted: 20 March 2016 . Accepted: 18 July 2016 . Published Online: 22 February 2017

Ralf Banisch, and Péter Koltai



But is geometry the *best* lens we can use to classify data according to underlying differences in dynamics?

Introduction

- What is phase space topology and why is it important?

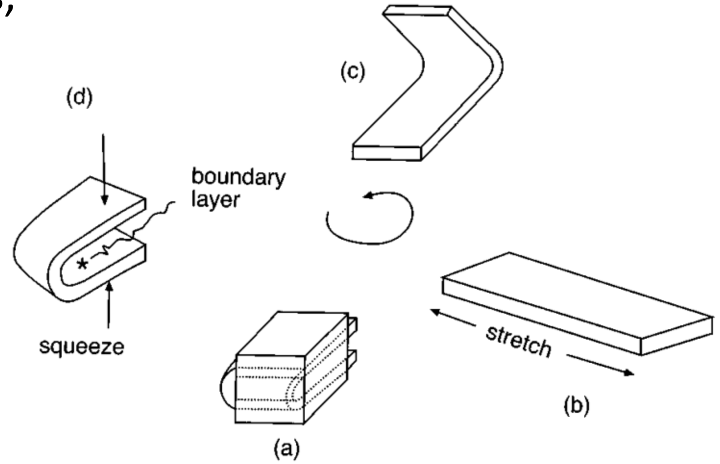
Invariants in phase space can be of different types:

a) Metric: dimensions of various types, e.g., correlation dimension (Grassberger & Procaccia, 1983), multifractal scaling functions (Halsey et al., 1986).

b) Dynamic: Lyapunov exponents (Oseledec, 1968; Wolf et al., 1985), as discussed by Eckmann & Ruelle (1985) and by Abarbanel et al. (1993).

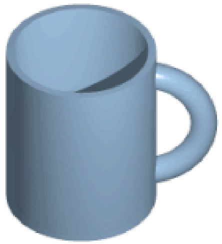
c) Topological: linking numbers, relative rotation rates, Conway polynomials, Branched Manifolds (Birman & Williams, 1983).

Invariants (a) and (b) do not provide information on how to model the system's dynamics, while (c) actually does!



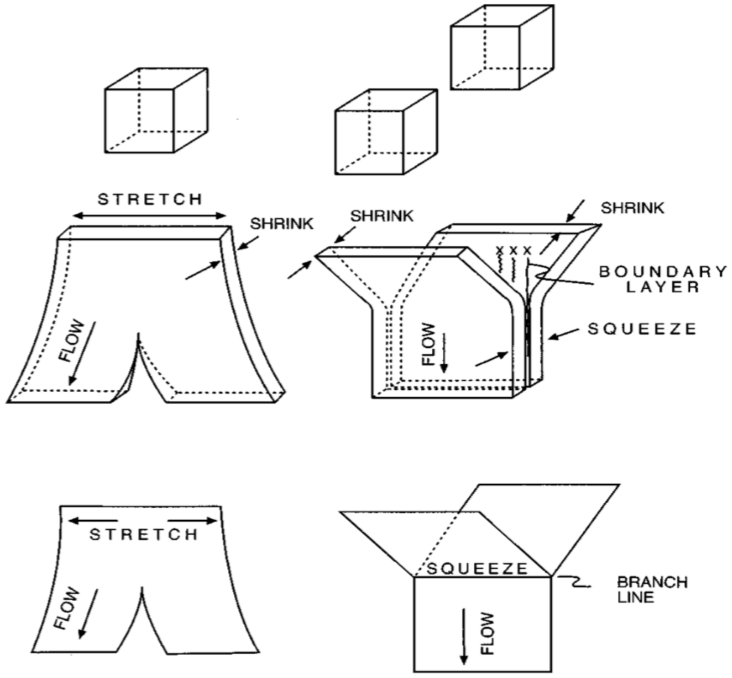
R. Gilmore, *Reviews of Modern Physics*, Vol. 70(4), October 1998

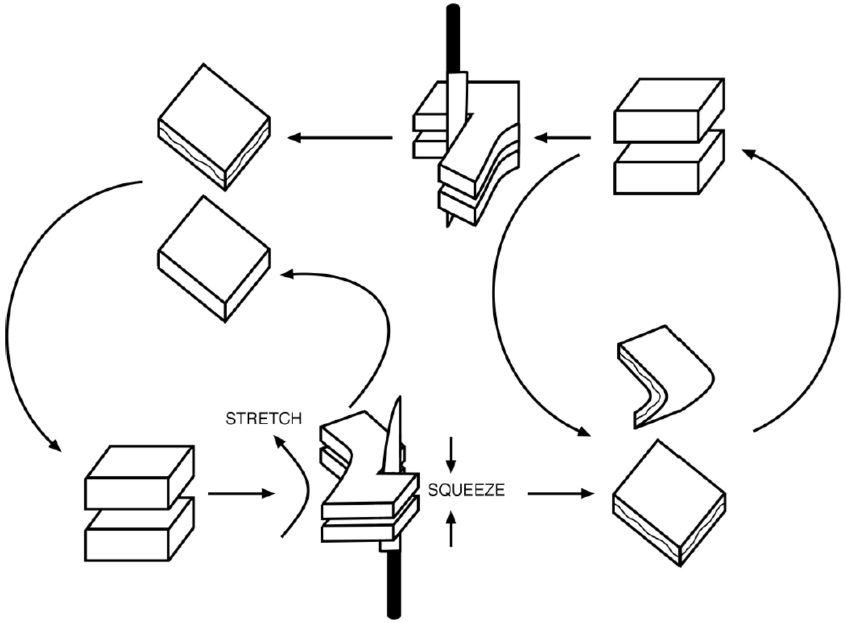
Introduction



- What is phase space topology and why is it important?

Topology is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling, and bending; that is, without closing holes, opening holes, gluing, or passing through itself. The animated image shows a continuous deformation of a mug into a doughnut: both objects are topologically equivalent.





Gilmore & Lefranc. *The Topology of Chaos: Alice in Stretch and Squeezeland*. Wiley-Interscience, 2002.

The “recipe” to “knead” the Lorenz’63 attractor is a sequence of steps that are topological in nature.

Introduction

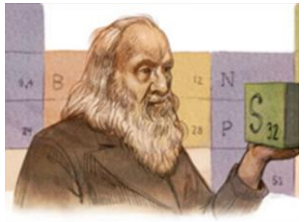
- What is phase space topology and why is it important?

The advantage of using topology, instead of geometry or fractality, to describe a flow in phase space lies in the fact that topology provides information about the invariant mechanisms that act in phase space to shape the flow.

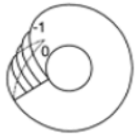
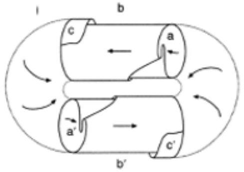
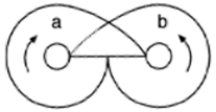
Geometry may differ, but if the underlying dynamics is equivalent, the topology should be the same.

Unveiling the topology <=> Unveiling the dynamics

Methods



- A table of elements for different types of dynamical behaviour ?

Dynamical system	ODEs	Parameters	Topologies
Rossler	$\dot{x} = -y - z$ $\dot{y} = x + ay$ $\dot{z} = b + z(x - c)$	$(a, b, c) = (2.0, 4.0, 0.398)$	
Duffing	$\dot{x} = y$ $\dot{y} = -\delta y - x^3 + x + A\sin(\omega t)$	$(\delta, A, \omega) = (0.4, 0.4, 1.0)$	
van der Pol	$\dot{x} = by + (c - dy^2)x$ $\dot{y} = -x + A\sin(\omega t)$	$(b, c, d, A, \omega) = (0.7, 1.0, 10.0, 0.25, \pi/2)$	
Lorenz	$\dot{x} = -\sigma x + \sigma y$ $\dot{y} = Rx - y - xz$ $\dot{z} = -bz + xy$	$(R, \sigma, b) = (26.0, 10.0, 8/3)$	

Gilmore & Lefranc. *The Topology of Chaos: Alice in Stretch and Squeezeland*. Wiley-Interscience, 2002.

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Metodi matematici per la scienza del clima

Methods

- Branched manifolds, knot-holders, cell complexes, homologies

The concept of branched manifold, introduced by Robert F. Williams in 1974, was anticipated in Edward Lorenz's famous 1963 paper: on page 20, he remarks that the trajectory 'lives' on a surface and describes the architecture of the attractor in terms of "isopleths."

EXPANDING ATTRACTORS

by R. F. WILLIAMS

TABLE OF CONTENTS

Publ. Math. IHES, v. 43 (1974), p. 169–203

Deterministic Nonperiodic Flow[1]

Edward N. Lorenz

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

138 JOURNAL OF THE ATMOSPHERIC SCIENCES VOLUME 20

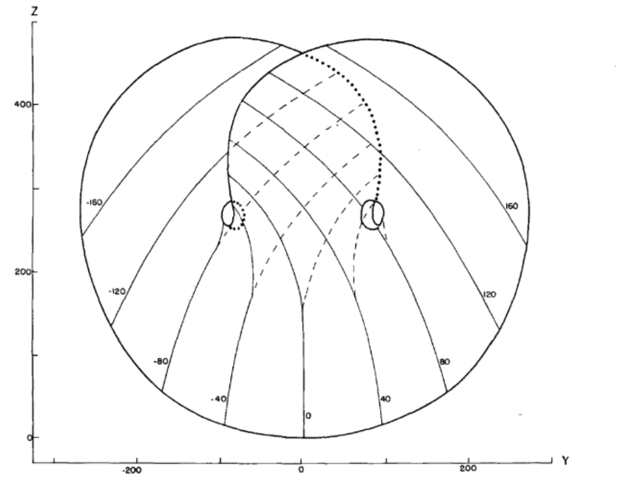
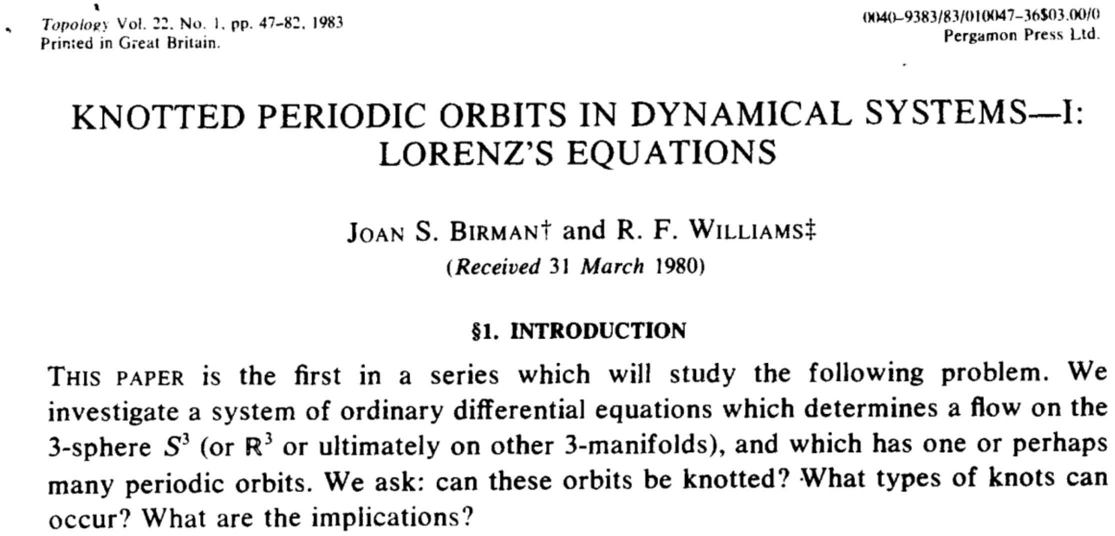


Fig. 3. Isopleths of X as a function of Y and Z (thin solid curves), and isopleths of the lower of two values of X, where two values occur (dashed curves), for approximate surfaces formed by all points on limiting trajectories. Heavy solid curve, and extensions as dotted curves, indicate natural boundaries of surfaces.

Methods

- Branched manifolds, knot-holders, cell complexes, homologies

Joan Birman & Robert F. Williams used branched manifolds to classify chaotic attractors in terms of the way unstable periodic orbits are “knotted” in dynamical systems.



Topology Vol. 22, No. 1, pp. 47–82, 1983
Printed in Great Britain.

0040–9383/83/010047–36$03.00/0
Pergamon Press Ltd.

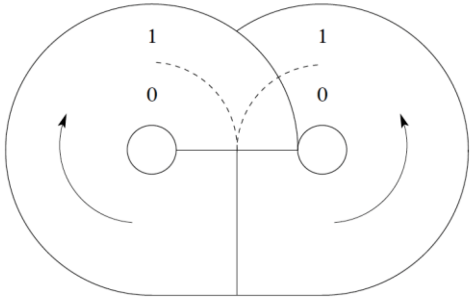
KNOTTED PERIODIC ORBITS IN DYNAMICAL SYSTEMS—I: LORENZ'S EQUATIONS

JOAN S. BIRMAN† and R. F. WILLIAMS‡

(Received 31 March 1980)

§1. INTRODUCTION

THIS PAPER is the first in a series which will study the following problem. We investigate a system of ordinary differential equations which determines a flow on the 3-sphere S^3 (or $\mathbb{R}^3$ or ultimately on other 3-manifolds), and which has one or perhaps many periodic orbits. We ask: can these orbits be knotted? What types of knots can occur? What are the implications?



$$x \simeq y \quad \text{if}$$

$$\lim_{t\to\infty} |x(t) - y(t)| = 0$$

The set of unstable periodic orbits (UPOs) of the Lorenz attractor lie on a “Branched Manifold”, i.e. on a structure obtained by identifying all the points with the same future (Birman-Williams projection). This is a 2-manifold almost everywhere (not where the flow splits or squeezes together).

Birman & Williams discovered that branched manifolds are a suitable concept to distinguish attractors which are not dynamically equivalent.

Methods

- Branched manifolds, knot-holders, cell complexes, homologies

In the late ‘90s, it was possible to determine whether two three-dimensional (3-D) dissipative dynamical systems are equivalent by using knot theory or templates.

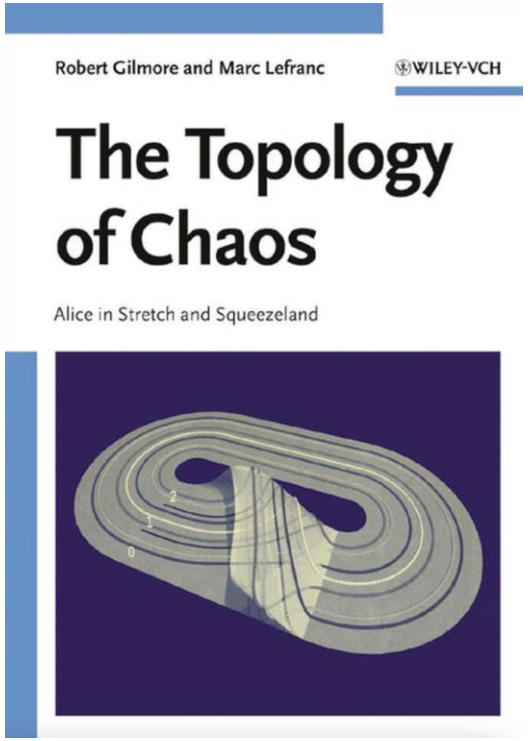
Topological analysis of chaotic dynamical systems

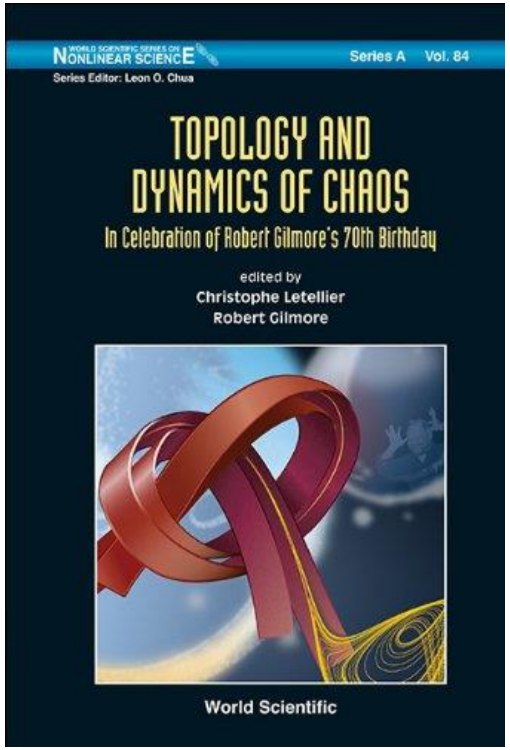
Robert Gilmore

Department of Physics & Atmospheric Science, Drexel University, Philadelphia, Pennsylvania 19104

Topological methods have recently been developed for the analysis of dissipative dynamical systems that operate in the chaotic regime. They were originally developed for three-dimensional dissipative dynamical systems, but they are applicable to all “low-dimensional” dynamical systems. These are systems for which the flow rapidly relaxes to a three-dimensional subspace of phase space. Equivalently, the associated attractor has Lyapunov dimension $d_L<3$. Topological methods supplement methods previously developed to determine the values of metric and dynamical invariants. However, topological methods possess three additional features: they describe how to model the dynamics; they allow validation of the models so developed; and the topological invariants are robust under changes in control-parameter values. The topological-analysis procedure depends on identifying the stretching and squeezing mechanisms that act to create a strange attractor and organize all the unstable periodic orbits in this attractor in a unique way. The stretching and squeezing mechanisms are represented by a caricature, a branched manifold, which is also called a template or a knot holder. This turns out to be a version of the dynamical system in the limit of infinite dissipation. This topological structure is identified by a set of integer invariants. One of the truly remarkable results of the topological-analysis procedure is that these integer invariants can be extracted from a chaotic time series. Furthermore, self-consistency checks can be used to confirm the integer values. These integers can be used to determine whether or not two dynamical systems are equivalent; in particular, they can determine whether a model developed from time-series data is an accurate representation of a physical system. Conversely, these integers can be used to provide a model for the dynamical mechanisms that generate chaotic data. In fact, the author has constructed a doubly discrete classification of strange attractors. The underlying branched manifold provides one discrete classification. Each branched manifold has an “unfolding” or perturbation in which some subset of orbits is removed. The remaining orbits are determined by a basis set of orbits that forces the presence of all remaining orbits. Branched manifolds and basis sets of orbits provide this doubly discrete classification of strange attractors. In this review the author describes the steps that have been developed to implement the topological-analysis procedure. In addition, the author illustrates how to apply this procedure by carrying out the analysis of several experimental data sets. The results obtained for several other experimental time series that exhibit chaotic behavior are also described. [S0034-6861(98)00304-3]

Reviews of Modern Physics, Vol. 70, No. 4, October 1998





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Metodi matematici per la scienza del clima

Methods

Computing topological invariants using **knots**…

3D trajectory set

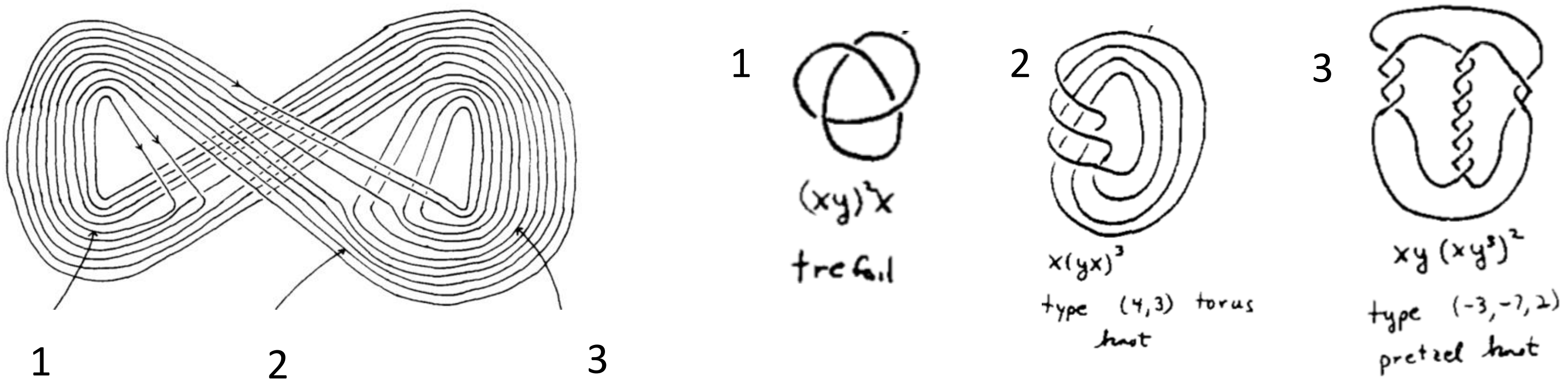
Knot invariants

- Branched manifolds, knot-holders, cell complexes, homologies

1) Approximate trajectories by closed curves.

2) Find a topological representation for the orbit structure.

3) Obtain an algebraic description for the topological structure.



J. S. Birman & R. F. Williams

Knotted periodic orbits in dynamical systems. Topology: Vol.22. No. I,pp.47~81. 1983

1) Close-returns method – time series, though, must be long and noise free.

2) Knot theory – knot ≡ orbit in three dimensions.

3) Knot invariants – *e.g.*, linking numbers, Conway polynomials.

Methods

- Branched manifolds, knot-holders, cell complexes, homologies

Knot information for a chaotic attractor can be condensed in a knot-holder or template.

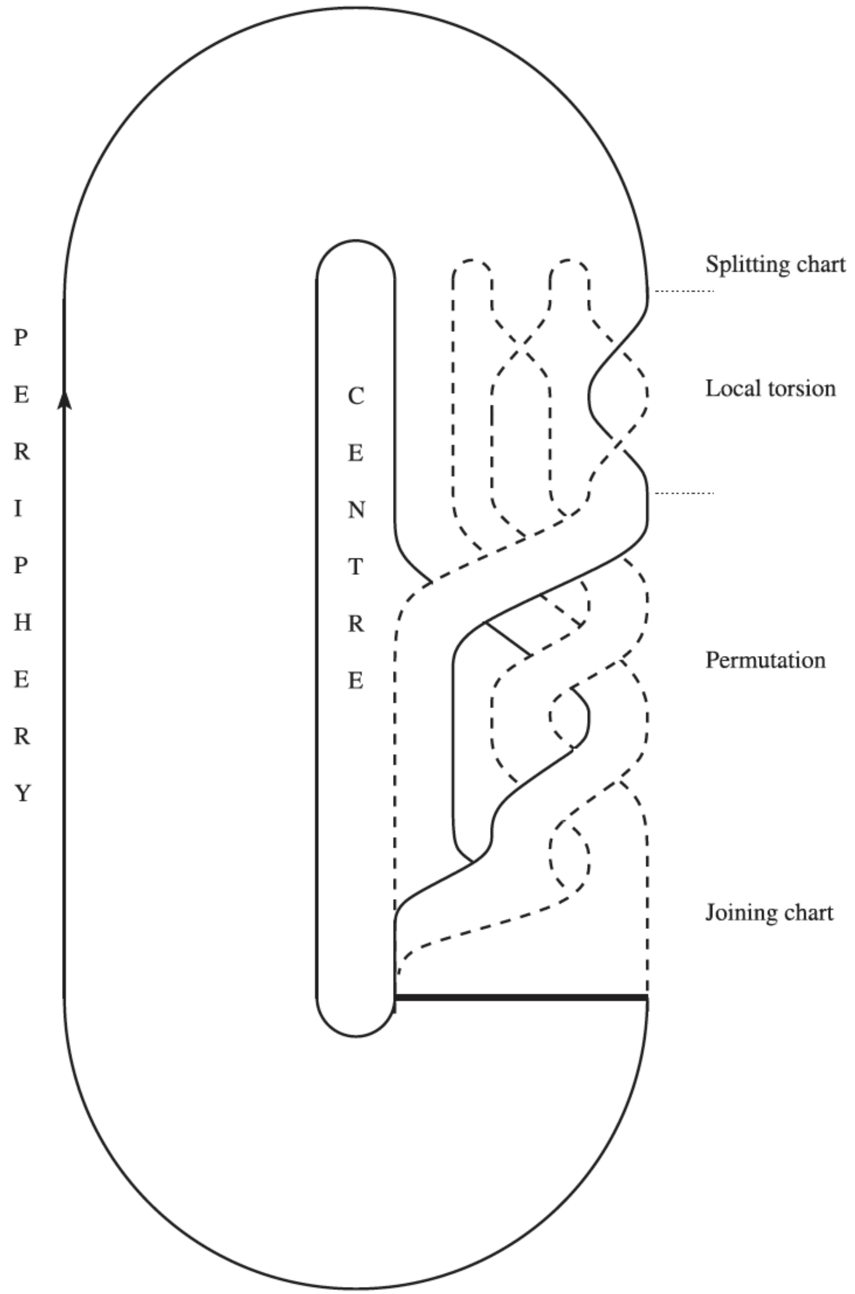
Definition

The template or knot-holder is a scheme used to assemble a set of strips that lodge the knots along the attractor.

Each strip represents a path followed by the flow.

⚠️ A template or knot-holder may require the introduction of fictitious boundaries between the different strips.

Typically, a strip is defined between and a joining chart & a splitting chart, ended by a joining line, which corresponds to a Poincaré section.



Methods

- Branched manifolds, knot-holders, cell complexes, homologies

What's wrong with templates?

Chapter 7

A braided view of a knotty story

Mario Natiello

Matematikcentrum-LTH, Lunds Universitet
Box 118, 221 00 Lund, Sverige

Hernán Solari

Departmento de Física, Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires, Argentina

Periodic orbits of 3-d dynamical systems admitting a Poincaré section can be described as braids. This characterisation can be transported to the Poincaré section and Poincaré map, resulting in the braid type. Information from braid types allows to estimate bounds for the topological entropy of the map while revealing detailed orbit information from the original system, such as the orbits that are necessarily present along with the given one(s) and their organisation. We review this characterisation with some examples –from a user-friendly perspective–, focusing on systems whose Poincaré section is homotopic to a disc.

There are several problems.

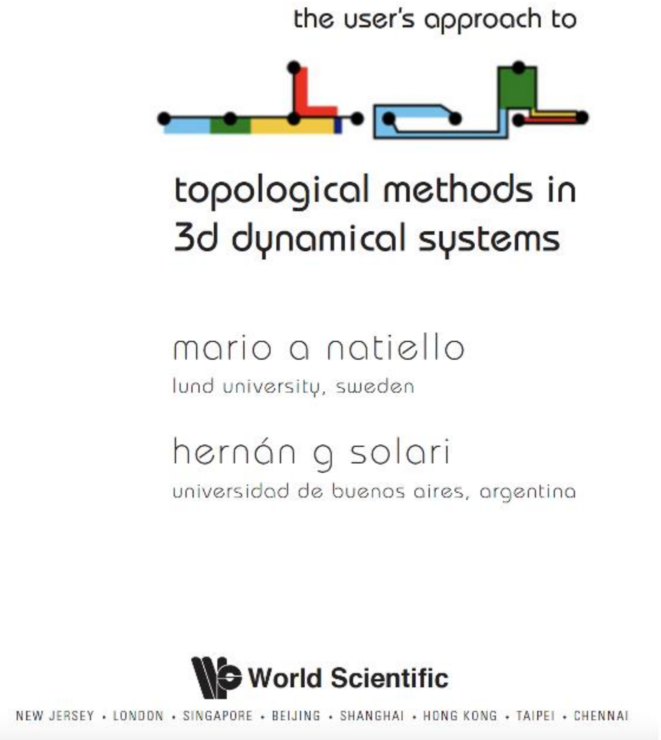
🚫 Reconstructing Unstable Periodic Orbits (UPOs) is not always possible.

🚫 The template is not necessarily topologically faithful to the branched manifold.

🚫 Templates are limited to 3D systems: knots unknot in higher dimensions.

6.6.3 ***Homology groups***

Still, we may want to understand the topological properties of the set of periodic orbits hidden in our data. We need some "braidless" method (in the sense that knots "dissolve" into trivial objects in higher dimensions) and one method that appears to jump at hand is to consider the homology groups associated to our data [Muldoon et al. 1993, Sciamarella and Mindlin 1999; 2001].



Methods

- Branched manifolds, knot-holders, cell complexes, homologies

Towards an "orbitless" and "knotless" methodology...

1993

Physica D 65 (1993) 1–16
North-Holland

SDI: 0167-2789(92)00026-1

PHYSICA D

Topology from time series

M.R. Muldoon[a], R.S. MacKay[a], J.P. Huke[b] and D.S. Broomhead[b]
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[b]*DRA at RSRE, Malvern, St. Andrew's Road, Great Malvern, Worcestershire WR14 3PS, United Kingdom*

Received 15 August 1992
Revised manuscript received 13 November 1992
Accepted 23 November 1992
Communicated by G. Ahlers

We describe methods for the study of topological pro
systems. We explain how to compute such invariants as the
and suggest a number of potential applications.

1999

VOLUME 82, NUMBER 7 PHYSICAL REVIEW LETTERS 15 FEBRUARY 1999

Topological Structure of Chaotic Flows from Human Speech Data

Denisse Sciamarella and G.B. Mindlin
Departamento de Física, FCEN, Universidad de Buenos Aires, Pab I, Ciudad Universitaria, cp 1428, Buenos Aires, Argentina
(Received 7 July 1998)

We report the analysis of branched manifolds through homolog
applicability of the topological approach to the analysis of human sp
cases are discussed. [S0031-9007(99)08424-0]

PACS numbers: 47.52.+j, 02.40.Sf, 43.72.+q

2001

PHYSICAL REVIEW E, VOLUME 64, 036209

Unveiling the topological structure of chaotic flows from data

Denisse Sciamarella and G. B. Mindlin
Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pab I, Ciudad Universitaria, Casilla de Correo 1428, Buenos Aires, Argentina
(Received 13 December 2000; published 21 August 2001)

We report the analysis of branched manifolds through homologies, in order to extend the range of applicability of the topological approach to the analysis of chaotic data. Analytic and numerical cases are discussed.

DOI: 10.1103/PhysRevE.64.036209 PACS number(s): 05.45.Pq, 47.52.+j, 02.40.Sf

Methods

- Branched manifolds, knot-holders, cell complexes, homologies

Computing topology using knot theory

3-D trajectory set

Knot invariants

RESTRICTIONS

- Precision and length of time series must be good enough for orbits in phase space to be reconstructed accurately …
- Phase space dimension *n* cannot be higher than three, since knots or braids unknot …

Computing topology using homologies

n-D point-cloud

Homologies

HOMOLOGY GROUPS

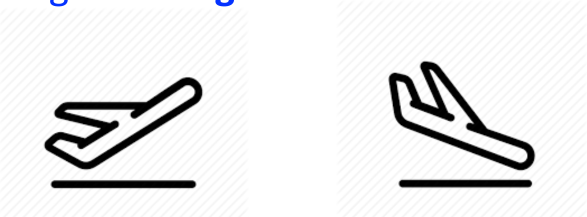
- Time series can be shorter and noisy since the method is independent of the reconstruction of trajectories in phase space (orbitless).
- Applicable in $n > 3$ dimensions: the method does not rely on knots or braids (knotless and braidless).

Methods

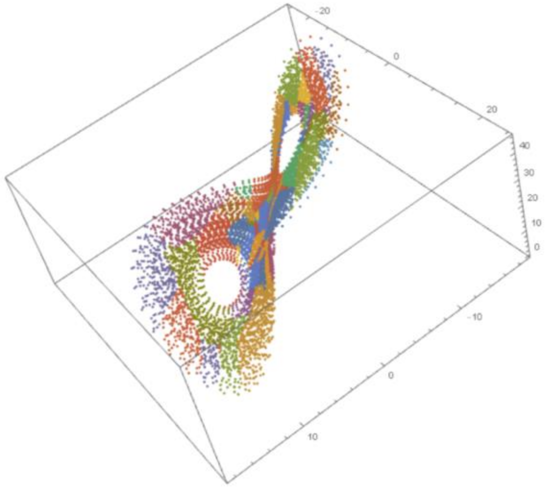
- Branched Manifold Analysis through Homologies (BraMAH)

1) Approximate points as lying on a branched manifold.
2) Find a topological representation for the branched manifold.
3) Obtain an algebraic description of the topological structure.

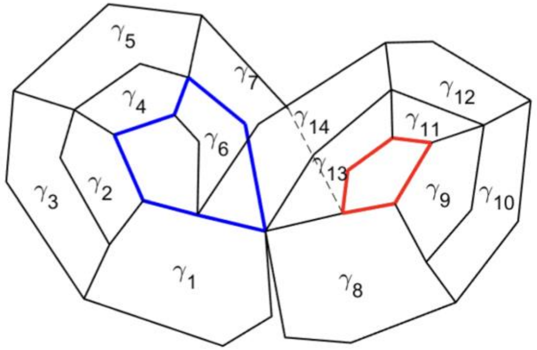
Computing topological invariants using **homologies**



n-D point-cloud → Homologies



BraMAH COMPLEX construction



HOMOLOGY GROUP computation

$$H_0 \approx Z^1;$$
$$H_1 \approx Z^2;$$
$$H_2 \approx 0$$

1) Local approximation by *d*-disks => short and noisy time series *can* be handled.
2) Build a cell complex keeping track of the gluing prescriptions => 3D+ *can* be handled.
3) Compute homologies and orientability properties of the cell complex => the structure can be identified.

Methods

- <u>Branched Manifold</u> Analysis through <u>Homologies</u> (BraMAH)

  Acronym first used in Charó, Artana & Sciamarella,
  Physica D 405 (2020) 132371

Sanskrit: ब्रह्मा, Romanized: Brahmā or Bramāh

Brahmā is a Hindu god, referred to as "the Creator" within the Trimurti, the trinity of supreme divinity. He is associated with knowledge and creation.

> A BraMAH cell complex is a particular type of cell complex, constructed to adjust to the topology of a branched manifold. Cells approximate sets of points lying on a branched manifold.

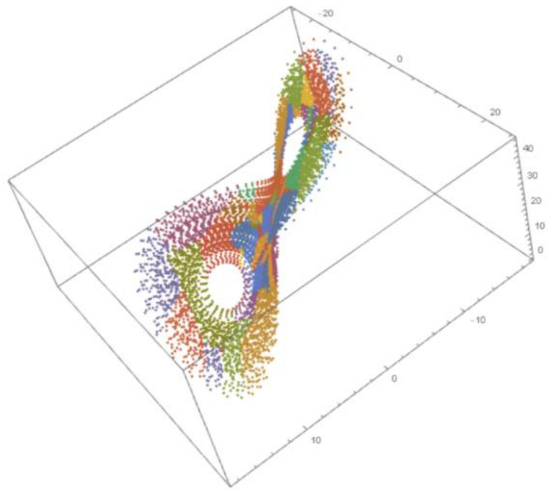
One of his hands holds mālā (Sanskrit: माला) symbolizing time.

Methods

- Branched Manifold Analysis through Homologies (BraMAH)

A **patch** is a set of points $\{x_i\}$ around an arbitrary point x_0, that is locally homeomorphic to the interior of a disk in d dimensions ($d \leq n$)

Point-cloud decomposition into patches

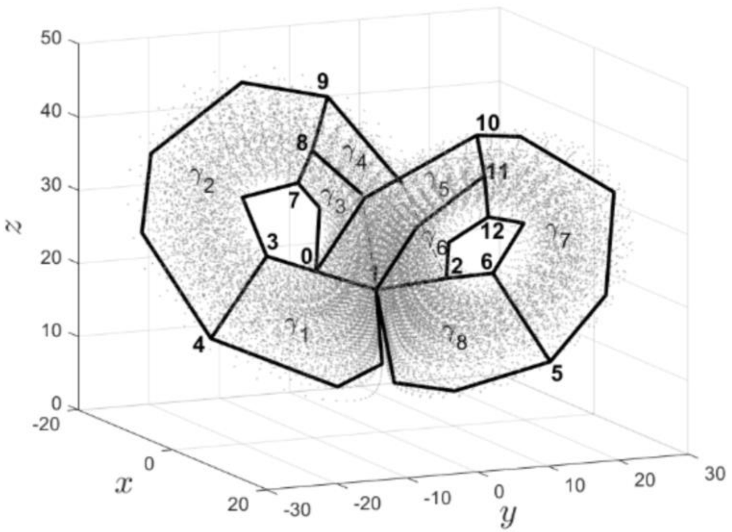


A patch is a good approximation of a hyperplane of dimension d in a space of dimension n if the square roots of d among the n second moments of $\{x_i\}$ decrease linearly with R while $(n–d)$ decrease as lower powers of R.

$X_{i,j} = (x_{i,j} - x_{0,j}) \Rightarrow$

find N_1 such that d among n singular values of $X_{i,j}$ vary linearly with N

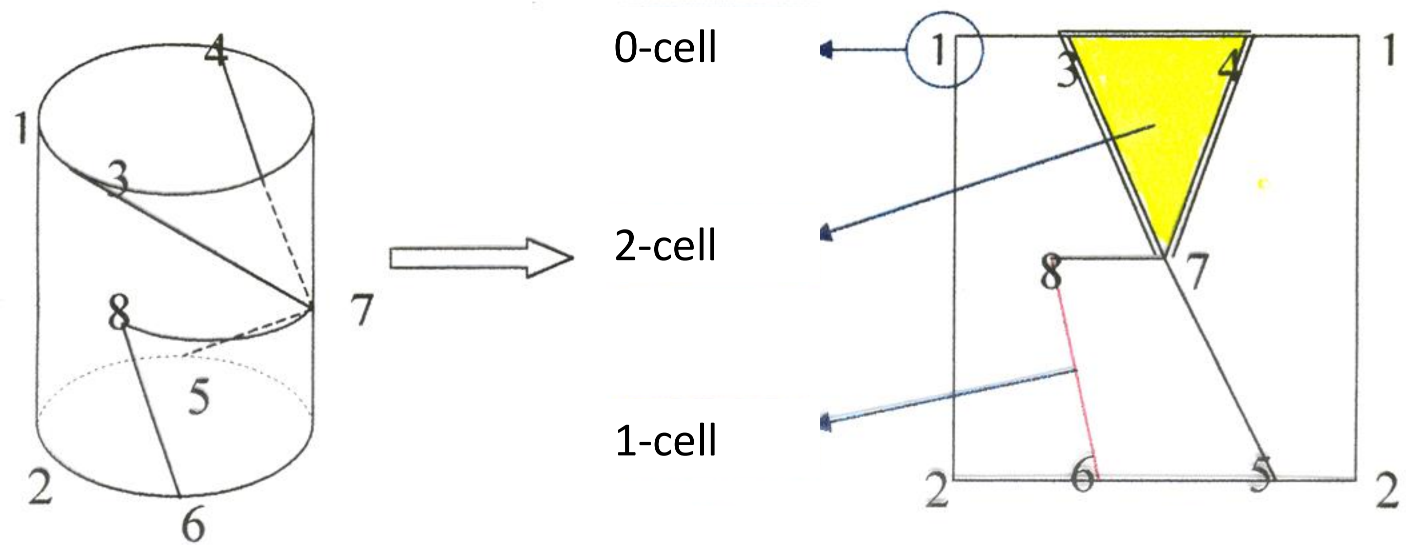
Patches are used to construct the cells in the cell complex



Methods

- Branched Manifold Analysis through Homologies (BraMAH)

How to determine if two spaces are topologically equivalent? → Cell complex covering the cylinder

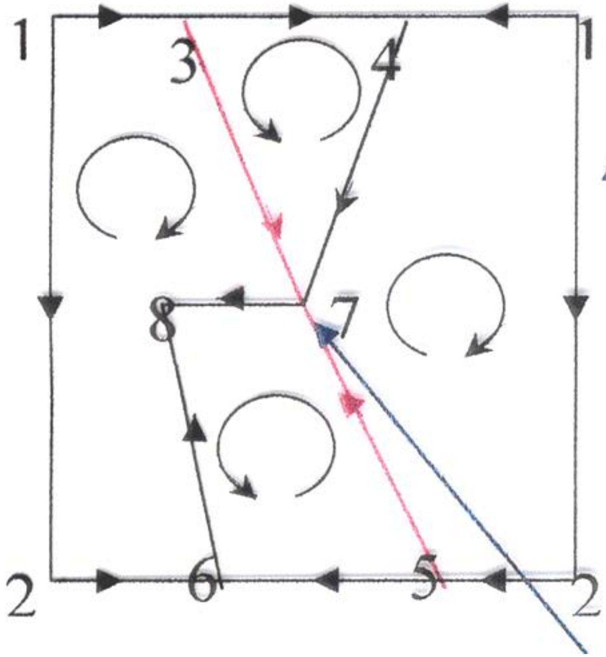


A ***k*-cell** is a set corresponding to the interior of a disk in *k* dimensions whose borders are divided into cells of lower dimension.

A **cell complex** is a set of cells such that their borders are elements of the complex with interiors that do not intersect.

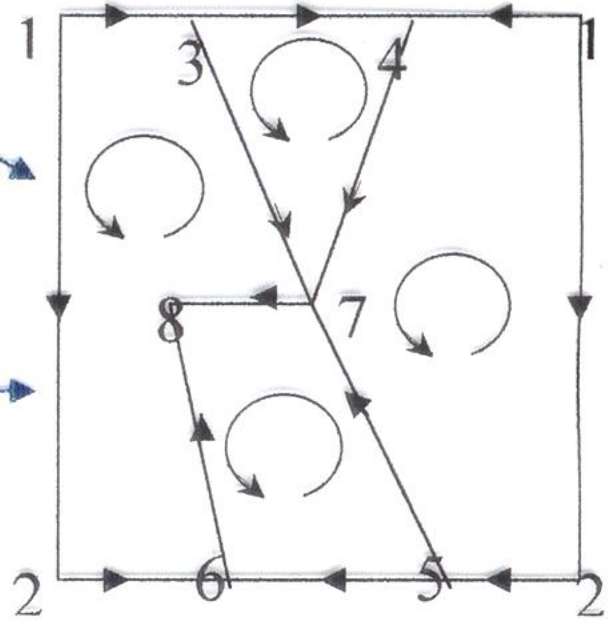
Methods

- Branched Manifold Analysis through Homologies (BraMAH)



Oriented complexes

Uniformly oriented complex



Example of a *1*-chain:
<3,7>-<5,7>

A ***k*-chain** in a complex K is a sum $C = \Sigma a_i \sigma_i$ such that σ_i are the k-cells with $a_i \in Z$ and such that $C_k(K)$ = {k-chains of K} has an abelian group structure.

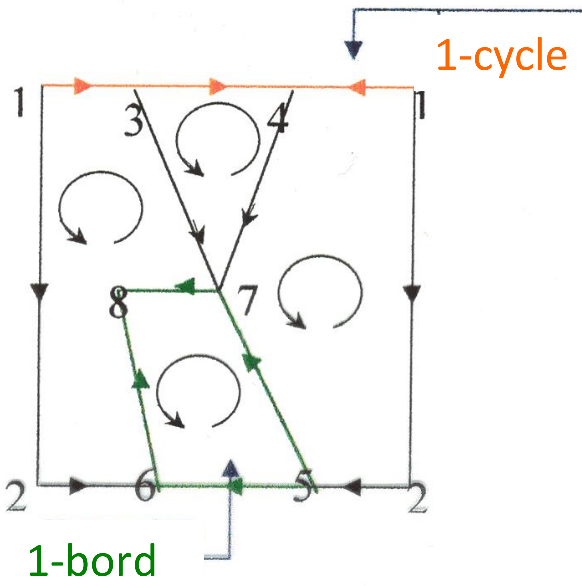
Example of border map:

∂(<3,7,4>)=<3,7>-<4,7>-<3,4>

A **border map** is an operation ∂: $C_k(K) \rightarrow C_{k-1}(K)$ such that $\partial(\Sigma a_i \sigma_i) = \Sigma a_i \, \partial(\sigma_i)$

Methods

- Branched Manifold Analysis through Homologies (BraMAH)

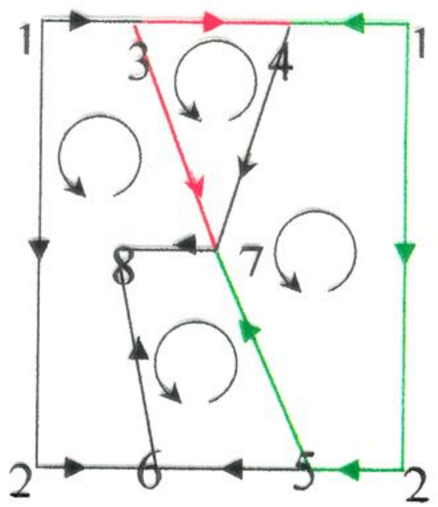


A ***k*-cycle** is a *k*-chain *C* such that $\partial(C)=0$

$Z_k(K)$ = {all *k*-cycles in a complex}

A ***k*-border** is a *k*-chain *C* / there exists a (*k*+1)-chain *D* such tat $\partial(D)=C$

$B_k(K)$ = {all *k*-borders of an *n*-complex}



Equivalence relationship:

Two *k*-chains C_1 and C_2 are called <u>homologically equivalent</u> ($C_1 \sim C_2$) if there exists a (*k*+1)-chain *D* such that $\partial(D)=C_1-C_2$

Example: -<3,4> + <3,7> ~ - <1,4> + <1,2> + <2,5> + <5,7>

Methods

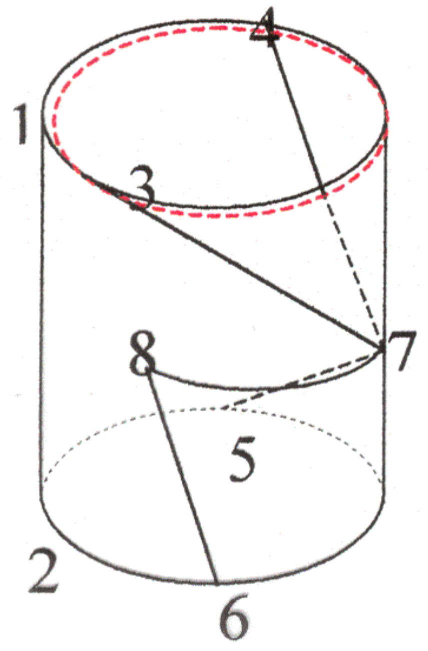
- Branched Manifold Analysis through Homologies (BraMAH)

Homology groups

The n+1 homology groups of an n-complex K are the sets:

$H_k(K) = Z_k/B_k$ ={the k-cycles being homologically independent that are not borders of any (k+1)-cell}

Example: the cylinder



K_1

$H_0(K_1)$=[[<1>]] $\sim Z^1 \Rightarrow$ one connected component

$H_1(K_1)$=[[<1,3>+<3,4>-<1,4>]] $\sim Z^1 \Rightarrow$ one nontrivial loop

$H_2(K_1)= \varnothing \sim 0 \Rightarrow$ **no cavities enclosed**

Methods

- Branched Manifold Analysis through Homologies (BraMAH)

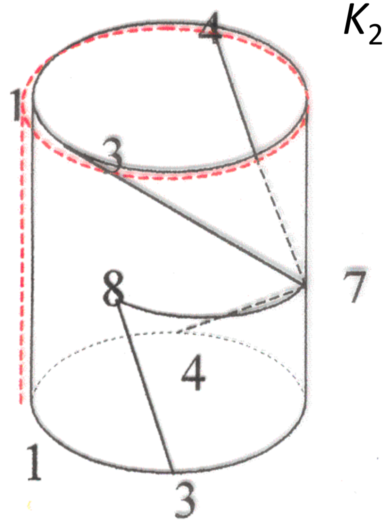
Homology groups

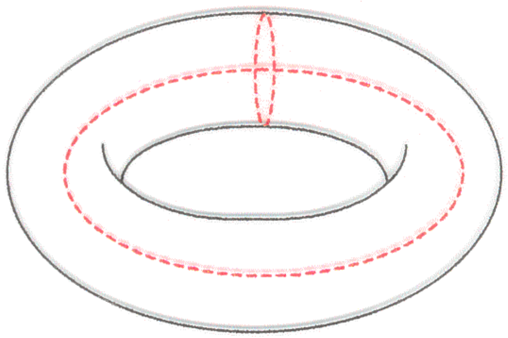
The *n+1* homology groups of an *n*-complex *K* are the sets:

$H_k(K) = Z_k/B_k$ ={the homologically independent *k*-cycles that are not borders of any (*k*+1)-cell}

Example: the torus

K_2





$H_0(K_2) = [[\langle 1 \rangle]] \sim Z^1 \Rightarrow$ one connected component

$H_1(K_2) \sim Z^2 \Rightarrow$ **two** nontrivial loops

$H_2(K_2) \sim Z^1 \Rightarrow$ **one cavity** enclosed

Methods

- Branched Manifold Analysis through Homologies (BraMAH)

Homology groups

The *n+1* homology groups of an *n*-complex *K* are the sets:

$H_k(K) = Z_k/B_k$ ={the homologically independent *k*-cycles that are not borders of any (*k*+1)-cell}

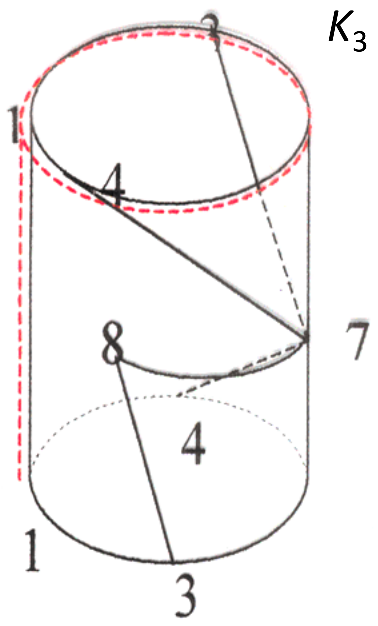
<u>Example</u>: the Klein bottle

$H_0(K_3) = [[\langle 1 \rangle]] \sim Z^1 \Rightarrow$ one connected component

$H_1(K_3) \sim Z^2 \Rightarrow$ two nontrivial loops

$H_2(K_3) \sim 0 \Rightarrow$ **no cavity** enclosed

The Klein bottle has a torsioned 1-cycle that is not the boundary of any 2-chain, but that becomes one if travelled twice, thus defining a **weak boundary**.



Methods

- Branched Manifold Analysis through Homologies (BraMAH)

An orientability chain in a uniformly oriented complex K with cells b_i is a chain $O = \partial(\Sigma b_i) = \Sigma\, a_j t_j$ if there exists at least one coefficient j such that $|a_j| > 1$. We call torsion chains the consecutive cells t_j preceded by the same multiple in O.

Example: Möbius strip.

$H_0(K_4)$ =[[<1>]]$\sim Z^1 \Rightarrow$ one connected component

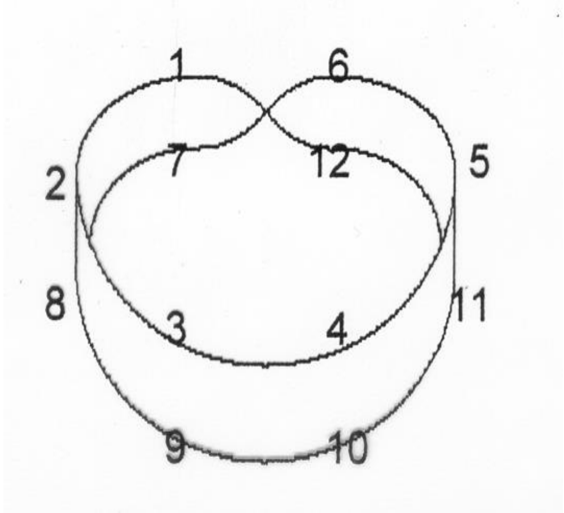
$H_1(K_4) \sim$ [[L]] $Z^1 \Rightarrow$ one nontrivial loop

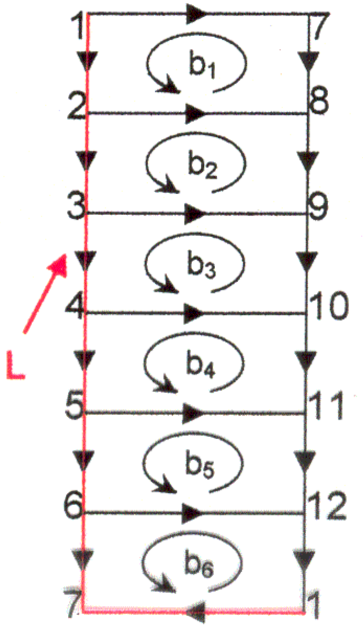
$H_2(K_4) \sim 0 \Rightarrow$ no cavities enclosed

$O(K_4) = \partial(\Sigma b_i)$ = -2 <1,7>

$T(K_4)$ ={<1,7>} $\Rightarrow$ one torsion located at <1,7>.

K_4





Metodi matematici per la scienza del clima

Methods

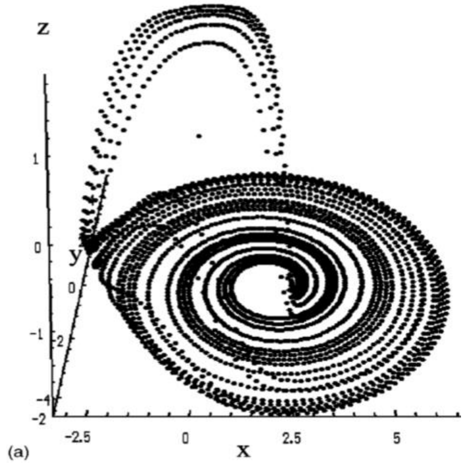
- Branched Manifold Analysis through Homologies (BraMAH)

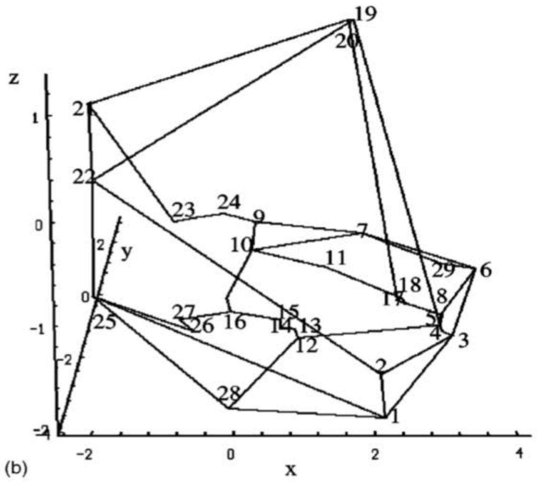
3D

$$x' = -(z+2)\, d\,(x-a) + (2-z)\,(\alpha(x-2) - \beta y - \alpha (x-2)\,((x-2)^2 + y^2 / R^2))$$

$$y' = -(z+2)\,(y-b) + (2-z)\,(\beta (x-2) - \alpha y - \alpha y\,(x-2)\,((x-2)^2 + y^2 / R^2))$$

$$\varepsilon z' = (4-z^2)\,(z+2 - m\,(x+2)) - \varepsilon c z$$





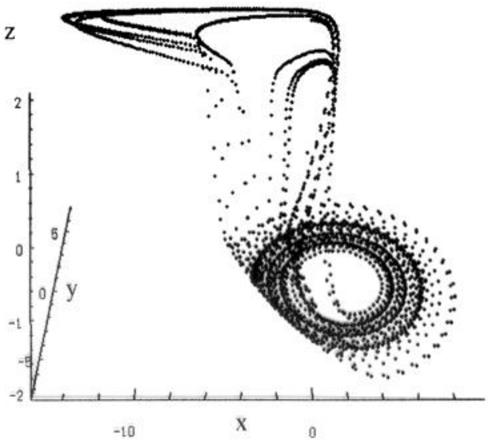
4D

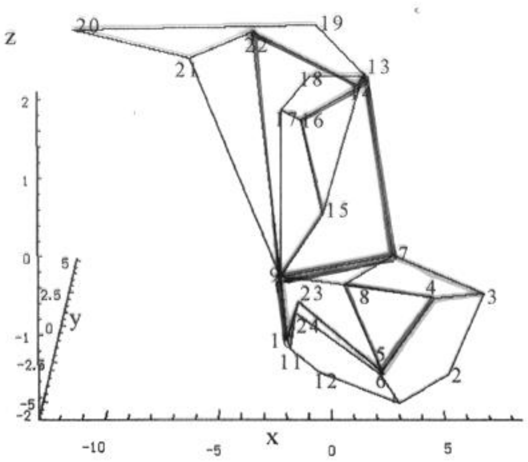
$$x' = -(z+2)\, d\,(x - (a + \varepsilon_3 (2+w))) + (2-z)\,(\alpha(x-2) - \beta y - \alpha (x-2)\,((x-2)^2 + y^2 / R^2))$$

$$y' = -(z+2)\,(y-b) + (2-z)\,(\beta (x-2) - \alpha y - \alpha y\,(x-2)\,((x-2)^2 + y^2 / R^2))$$

$$\varepsilon z' = (4-z^2)\,(z+2 - m\,(x+2)) - \varepsilon_1 c z$$

$$\varepsilon w' = (4-z^2)\,(z+2 - m\,(x+2)) - \varepsilon_2 c z$$

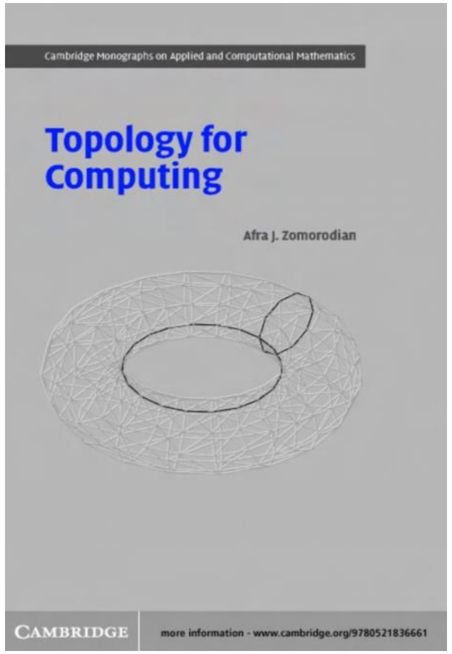


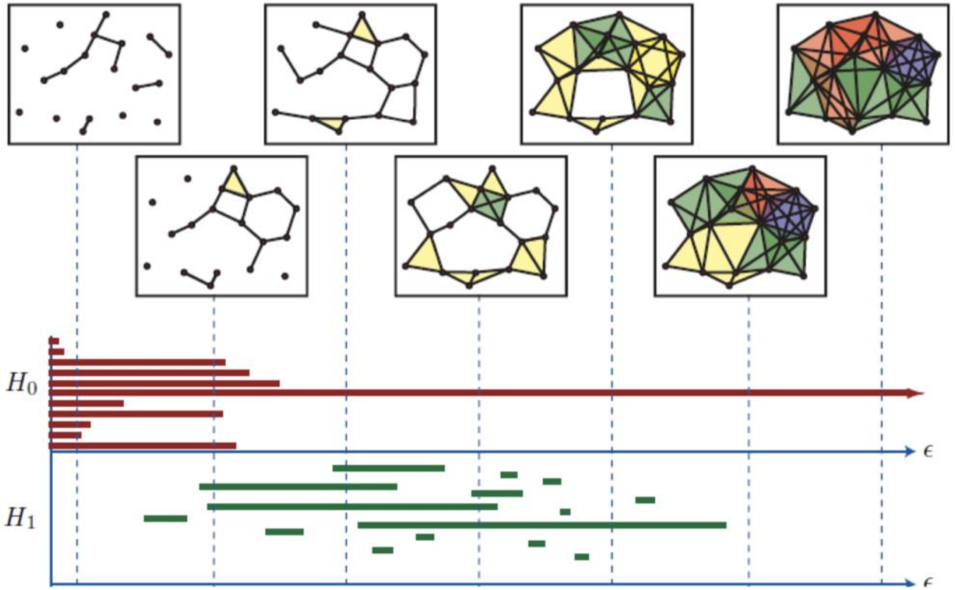


Methods

- Persistent homologies

The concept of persistent homology (PH) emerged independently in Bologna, in Colorado, and within a bio-geometry project in North Carolina, towards 2005.





The method constructs a series of cell complexes using a rule that depends on a distance parameter (ε or *d*).

The connectivity of the point cloud increases as ε or *d* grows.

Zomorodian, A. J. *Topology for computing* (Vol. 16). Cambridge University Press.

PH was conceived to solve <u>pattern recognition problems</u>, mainly in scanned images.

Methods

- Persistent homologies

Software package	Creator	Latest release	Release date	Software license[7]	Open source	Programming language
OpenPH	Rodrigo Mendoza-Smith, Jared Tanner	0.0.1	25 April 2019	Apache 2.0	Yes	Matlab, CUDA
javaPlex	Andrew Tausz, Mikael Vejdemo-Johansson, Henry Adams	4.2.5	14 March 2016	Custom	Yes	Java, Matlab
Dionysus	Dmitriy Morozov			GPL	Yes	C++, Python bindings
Perseus	Vidit Nanda	4.0 beta		GPL	Yes	C++
PHAT	Ulrich Bauer, Michael Kerber, Jan Reininghaus	1.4.1			Yes	C++
DIPHA	Jan Reininghaus				Yes	C++
Gudhi	INRIA	2.1.0	28 January 2018	GPLv3	Yes	C++, Python bindings
CTL	Ryan Lewis	0.2		BSD	Yes	C++
phom	Andrew Tausz				Yes	R
TDA	Brittany T. Fasy, Jisu Kim, Fabrizio Lecci, Clement Maria, Vincent Rouvreau	1.5	16 June 2016		Yes	R
Eirene	Gregory Henselman	1.0.1	9 March 2019	GPLv3	Yes	Julia
Ripser	Ulrich Bauer	1.0.1	15 September 2016	LGPL	Yes	C++
the Topology ToolKit	Julien Tierny, Guillaume Favelier, Joshua Levine, Charles Gueunet, Michael Michaux	0.9.2	25 June 2017	BSD	Yes	C++, VTK and Python bindings
Software package	**Creator**	**Latest Release**	**Release date**	**Software license[7]**	**Open source**	**Programming language**

Methods

- Persistent homologies

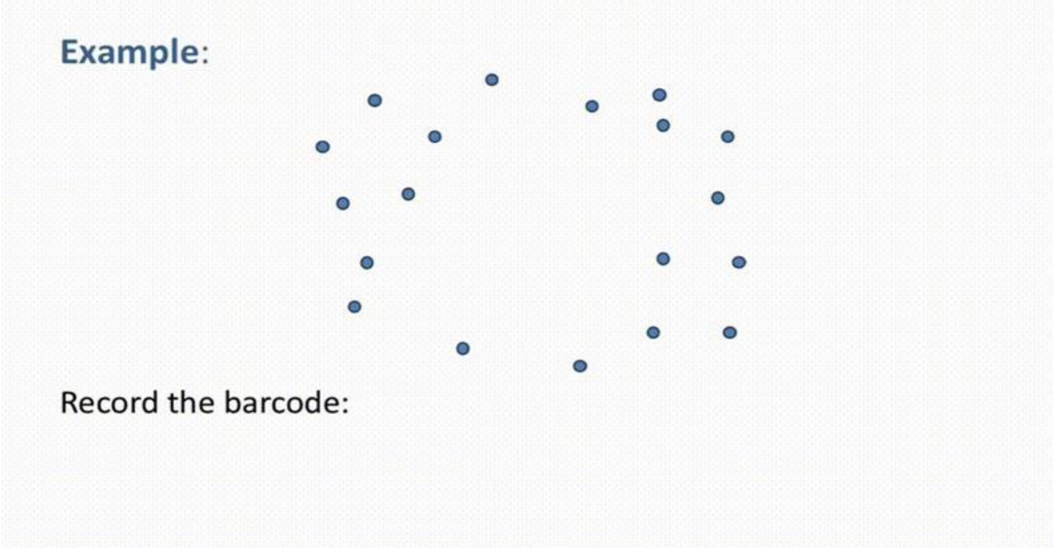
Filtrations: the rules used to build cell complexes as the filtration parameter is varied. The Vietoris-Rips filtration is illustrated in the gif below.

<u>Vietoris-Rips cell complex</u>

- A ball of diameter *d* is drawn around these points.

- Two balls intersect (two points are separated by a distance less than *d*) → connect the two points with a segment or 1-cell (simplicial cell of dimension 1).

- The triangles formed are completed by forming 2-cells (simplicial cells of dimension 2), and so on.



The Vietoris–Rips complex was originally called the Vietoris complex, for Leopold Vietoris, who introduced it as a means of extending homology theory from complexes to metric spaces.

Methods

- Persistent homologies

PH is not a Branched Manifold approximation method but *can* help counting holes in phase space point-clouds, serving as a guide.

Practical problems with PH

- The number of holes will depend on the choice of ε_{max} which is always somewhat arbitrary.
- H_k generators and cell complexes are generally not provided as output.

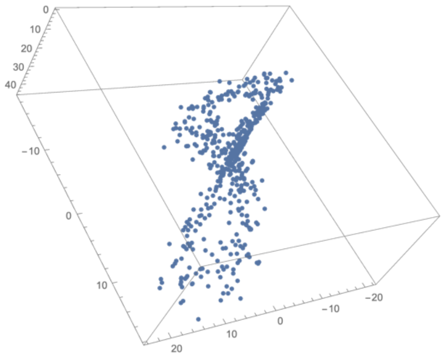
Fundamental problems with PH

The complexes are constructed in such a way that:

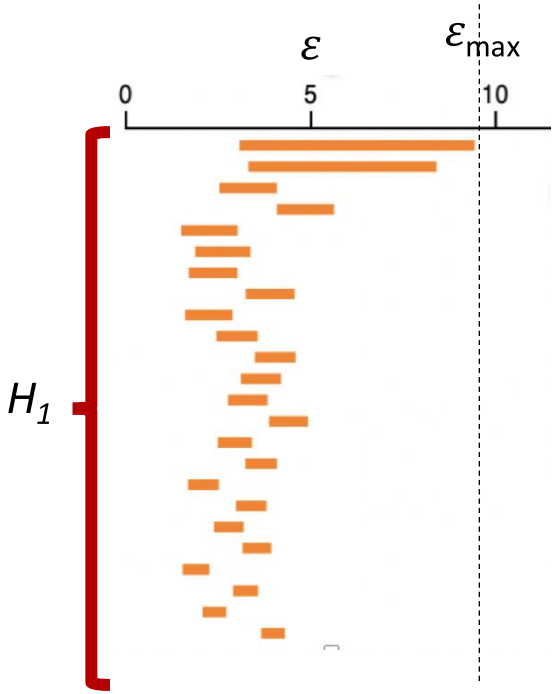
(1) #cells ≫ #points in the point-cloud ⇒ large point clouds are not supported.

(2) the complexes are not topologically faithful to the branched manifold.

Constructing the BraMAH complex from a Vietoris-Rips complex is an interesting open problem in computational topology.



https://live.ripser.org



Applications

- Topological methods can be harnessed for multiple purposes

> "Topological methods can be used to determine whether or not two dynamical systems are equivalent; in particular, they can determine whether a model developed from time-series data is an accurate representation of a physical system. Conversely, it can be used to provide a model for the dynamical mechanisms that generate chaotic data."
>
> R. Gilmore, Reviews of Modern Physics, Vol. 70, No. 4, October 1998

- ✓ Validate/refute models – simulations vs. observations.
- ✓ Comparing models – time series generated by different models.
- ✓ Comparing datasets – e.g., *in situ* versus satellite data.
- ✓ Extracting models from data – using global modeling techniques with a topological validation.
- ✓ Characterizing and labeling chaotic behaviors – towards a systematic classification.
- ✓ Classifying sets of time series according to their main dynamical traits – e.g., in Lagrangian Analysis.

Applications

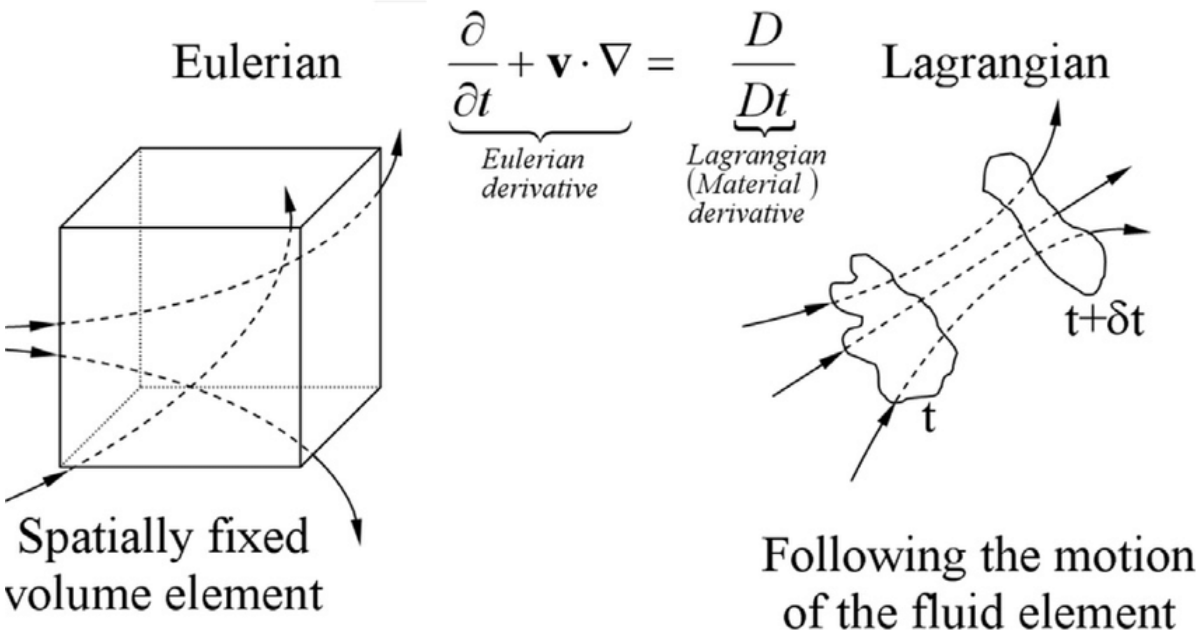
- Lagrangian Analysis

<u>What is Lagrangian analysis</u>?

In fluid mechanics, two viewpoints are possible.

In the Eulerian viewpoint, fluid motion is observed at specific locations in space, as time passes.

In the Lagrangian viewpoint, the observer follows individual fluid particles as they move through the fluid domain.



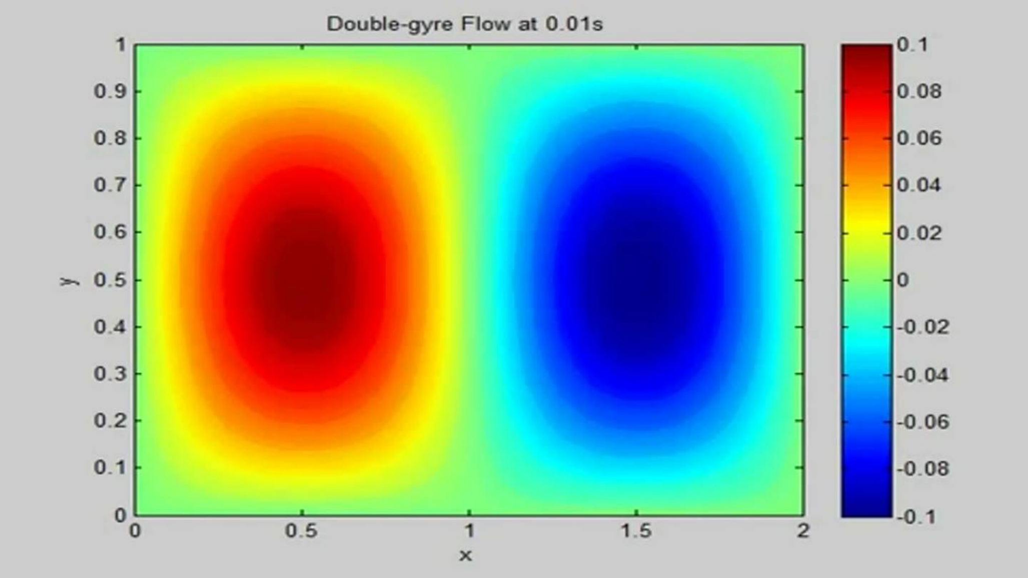
The Driven Double Gyre (DDG) system is an analytic model, often used to show how much Lagrangian patterns may differ from patterns in Eulerian fields.

It was introduced by Shadden et al. (2005) to mimic the motion of <u>two adjacent oceanic gyres enclosed by land</u> and, since Sulalitha Priyankara et al. (2017), it is known to present chaotic transport in some ‘regions’ of the fluid, even if the Eulerian picture is periodic.

Applications

- Lagrangian Analysis

Let us consider the kinematic model inspired in a pattern that occurs frequently in geophysical flow.



$$(x, y) \in D = [0, 2] \times [0, 1]$$

$$\begin{cases} \frac{dx(t)}{dt} = -\pi A \sin(\pi * forcing) \cos(\pi y) \\ \frac{dy(t)}{dt} = \pi A \cos(\pi * forcing) \sin(\pi y)(2ax + b); \end{cases}$$

$$a = \epsilon \sin(\omega t),\ b = 1 - 2\epsilon \sin(\omega t)$$

$$forcing = ax^2 + bx$$

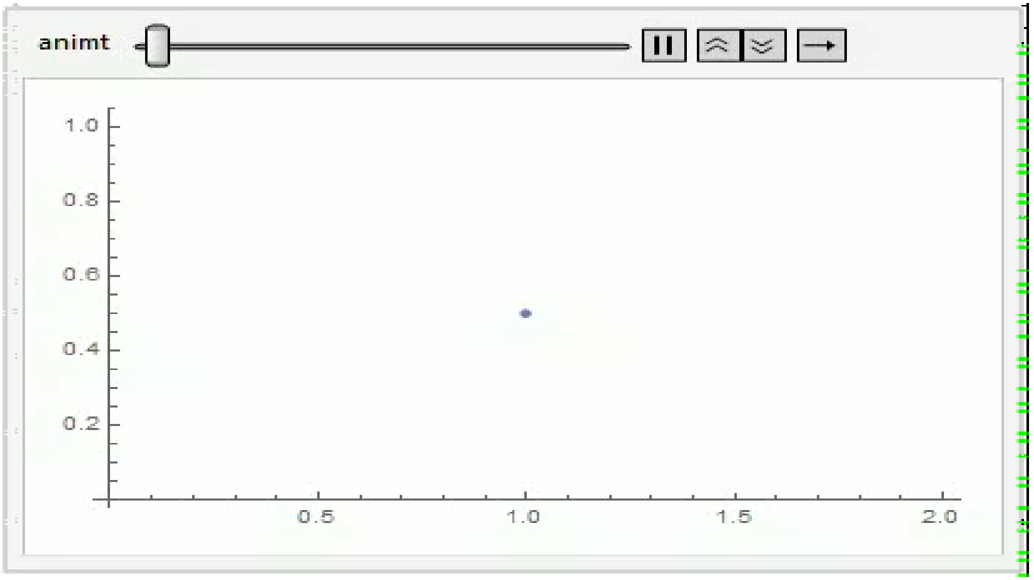
$$t_0 = 0,\ t_f = 10,\ \epsilon = 0.1,\ A = 0.1,\ \omega = \frac{\pi}{5}$$

From the Eulerian perspective the periodically driven Double-Gyre flow has a periodic and simple behaviour. But what about particle behaviour? What happens, for instance, if there is an "oil spill" in the middle of the domain?

Applications

- Lagrangian Analysis

Let us paint in blue the particles that are continuously passing through the centerpoint (streakline) to "visualize" particle behaviour.



Transport barriers appear, showing that the tracer invades some parts of the domain leaving some other regions blank.

These non-mixing islands move circularly in each half-domain.

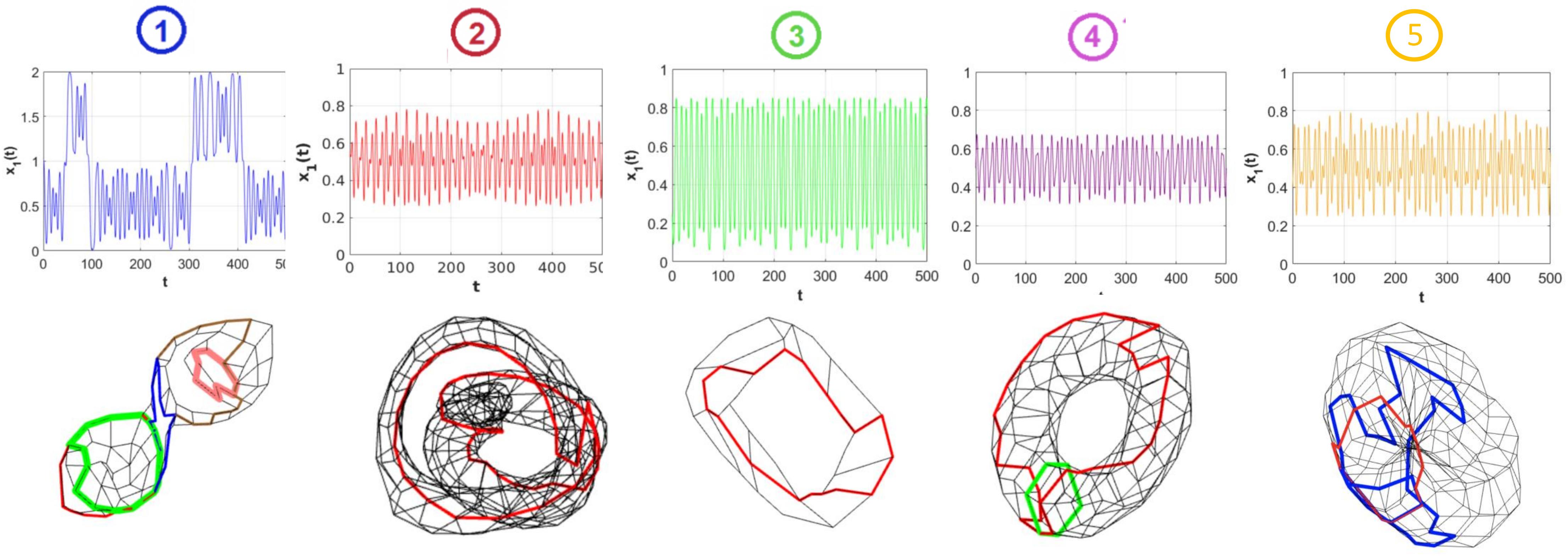
What can topology tell us in a problem of this kind?

Lagrangian time series (the position or the velocity of a particle) can be generated and studied in state space with our topological tools.

Applications

- Lagrangian Analysis

A BraMAH analysis applied to a collection of 8528 particles (x_1 time series) in a time window of 500 units yields five topological classes. Computations involve complexes constructed from 4-dimensional time-delay embeddings.



Five-loop structure | Moebius strip | Standard strip | Torus | Klein bottle

Applications

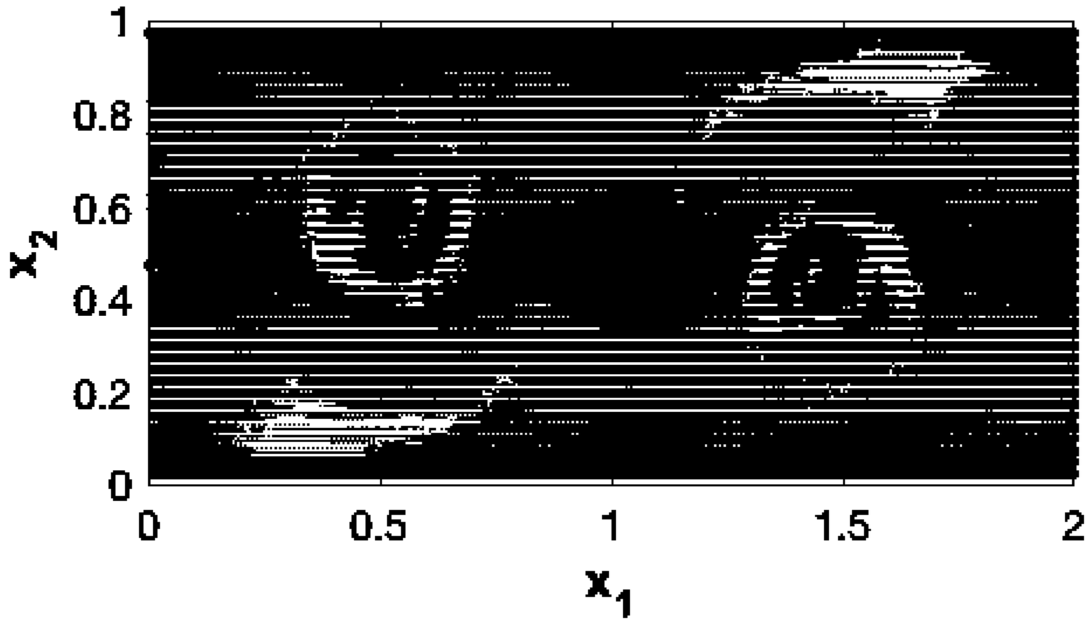
- Lagrangian Analysis

Topological colouring of 8528 advected particles in the driven Double-Gyre flow to visualize how topologies are organized in physical space.

Assigning a different color to each topological class, the colors in motion define particle sets that move together forming coherent regions, i.e., without mixing with the surrounding fluid.

Let us use the term 'separator' to designate the frontier between differently colored regions.

Such flow separators are associated with 'Lagrangian coherent structures', known to separate dynamically distinct regions in fluid flows (Kelley, Allshouse & Ouellette, 2013).



Applications

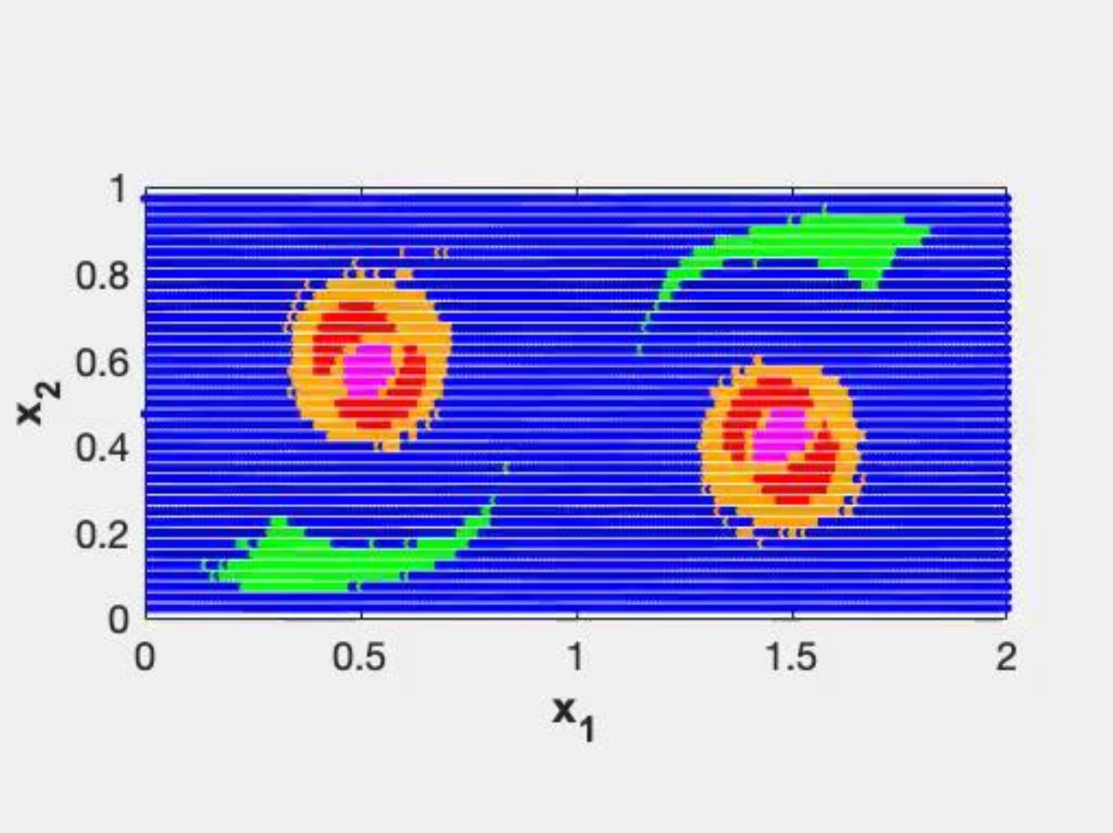
- Lagrangian Analysis

Topological colouring of 8528 advected particles in the driven Double-Gyre flow to visualize how topologies are organized in physical space.

If the advected particles are coloured according to the BraMAH topological analysis, the non-mixing islands become apparent.

Classifying topologies (= classifying dynamics) can be used:

i. for an indirect identification of particle sets that do not mix with the surrounding fluid;
ii. to characterize such dynamics within each region;
iii. to compare distant regions behaving similarly; and
iv. to compare the behaviour of particles in different flows.



$$a = \epsilon \sin(\omega t),\ b = 1 - 2\epsilon \sin(\omega t)$$

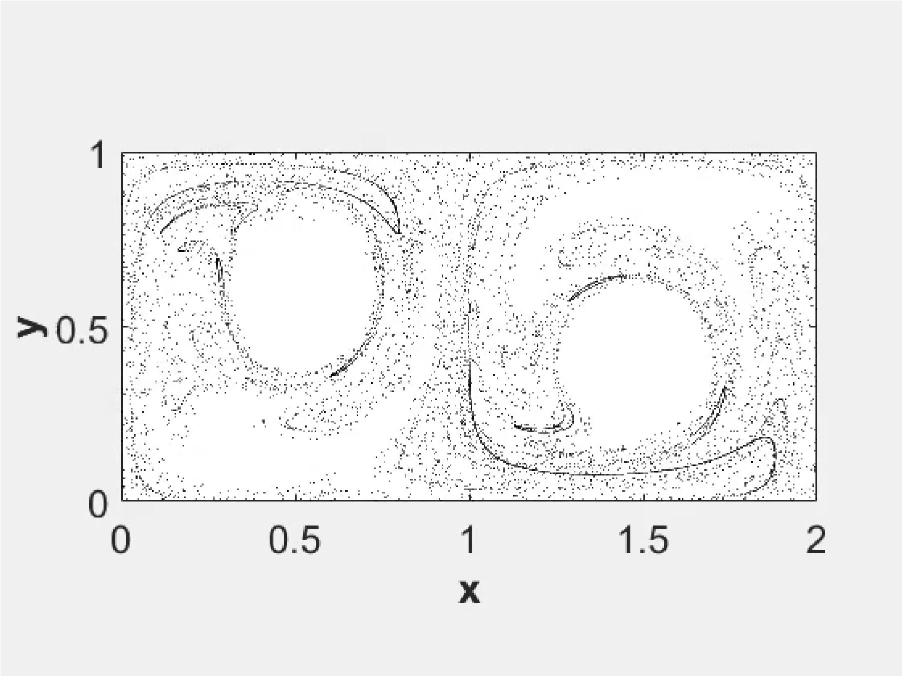
↓

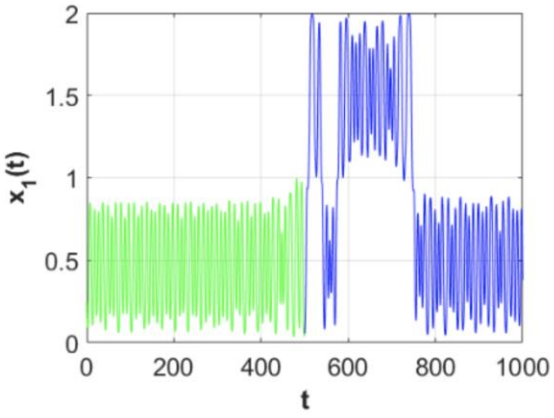
$$a(t) = \eta \sin\left[\omega t \left(1 + \gamma \sin(\frac{t}{2})\right)\right], b(t) = 1 - 2a(t)$$

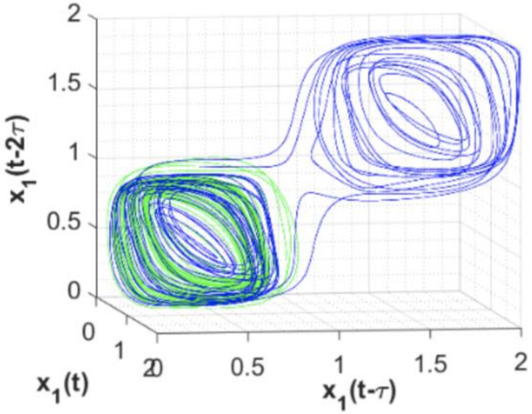
Applications

- Lagrangian Analysis

What happens if we introduce a perturbation in the driving force of the Double-Gyre in which the particles in the formerly non-mixing islands slowly migrate towards the chaotic sea?





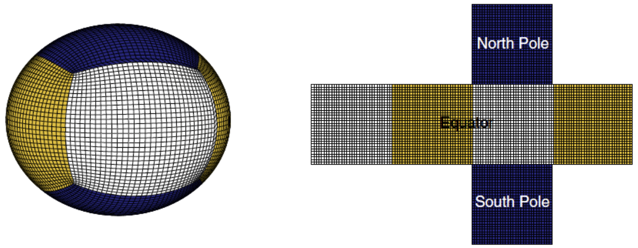


The topology that is computed is always referred to the time window that is chosen for the analysis. A particle that migrates 'moves' from one topological class to the other.

Metodi matematici per la scienza del clima

Applications

- Climate dynamics



Can homologies distinguish between simulated climate attractors?

Climate Dynamics
https://doi.org/10.1007/s00382-019-04926-7

Check for updates

Co-existing climate attractors in a coupled aquaplanet

M. Brunetti[1] · J. Kasparian[1] · C. Vérard[2]

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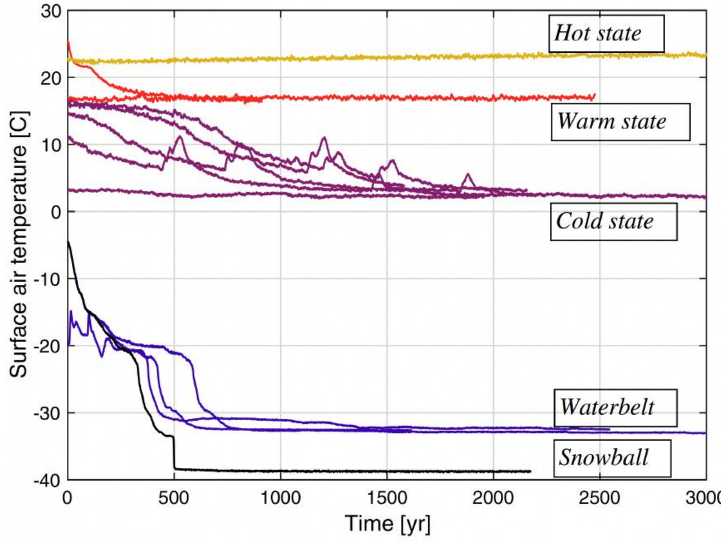
Abstract
The first step in exploring the properties of dynamical systems like the Earth climate is to identify the different phase space regions where the trajectories asymptotically evolve, called 'attractors'. In a given system, multiple attractors can co-exist under the effect of the same forcing. At the boundaries of their basins of attraction, small changes produce large effects. Therefore, they are key regions for understanding the system response to perturbations. Here we prove the existence of up to five attractors in a simplified climate system where the planet is entirely covered by the ocean (aquaplanet). These attractors range from a snowball to a hot state without sea ice, and their exact number depends on the details of the coupled atmosphere–ocean–sea ice configuration. We characterise each attractor by describing the associated climate feedbacks, by using the principal component analysis, and by measuring quantities borrowed from the study of dynamical systems, namely instantaneous dimension and persistence.

Keywords Coupled aquaplanet · Attractors · GCM · Complexity



Attractor	Waterbelt	Cold state	Warm state	Hot state
SAT (°C)	-33.00 ± 0.03	2.2 ± 0.1	17.0 ± 0.2	23.4 ± 0.1
Ocean temperature (°C)	-1.6449 ± 0.0008	3.225 ± 0.002	9.877 ± 0.004	17.418 ± 0.006
Sea ice extent (10^6 km^2)	429.51 ± 0.06	160.4 ± 0.9	67 ± 1	0
Latitude of sea ice boundary	9	43	60	90
TOA budget (W/m^2)	1.6 ± 0.1	3.0 ± 0.2	2.6 ± 0.2	2.3 ± 0.2
Ocean surface budget Q_{net} (W/m^2)	0.01 ± 0.14	-0.009 ± 0.217	0.03 ± 0.29	0.05 ± 0.27

https://web.dm.uba.ar/files/tesis_lic/2023/Salvagni.pdf

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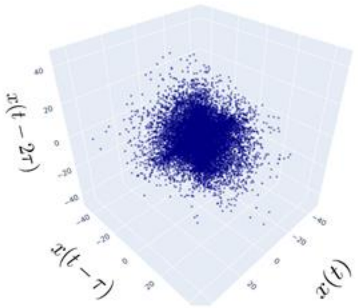
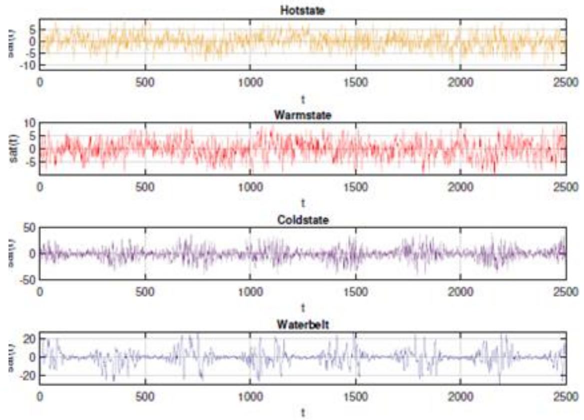
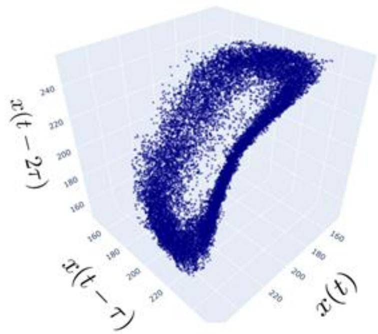
Applications

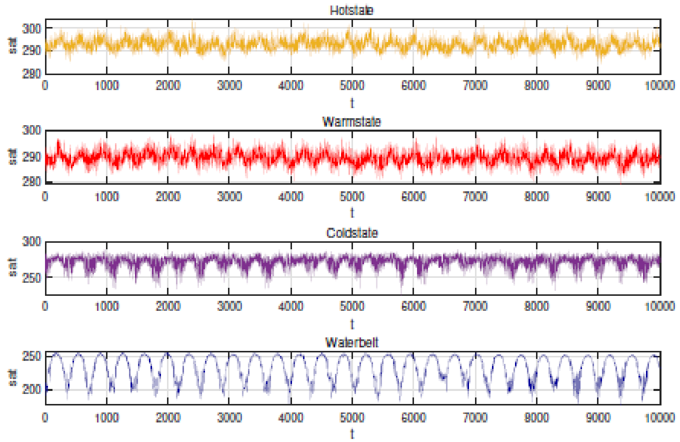
- Climate dynamics

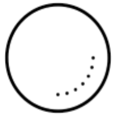
The time series of the annual averages are, when embedded, indistinguishable from each other, giving rise to point clouds distributed in the form of solid spheres or solid tori.

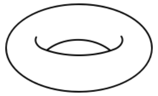












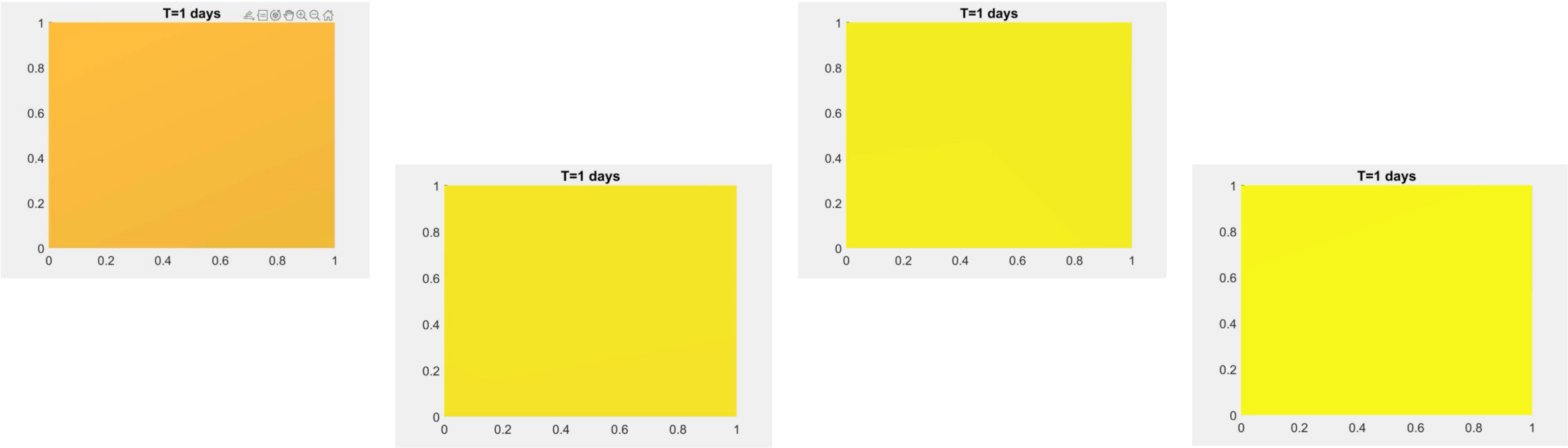
Solid sphere in phase space → statistical version of a fixed point in phase space: transient discarded, the system stabilizes around a given point.

Solid torus in phase space → When the predominant dynamics in the global variability is that of the seasonal cycle; see Falasca, F., & Bracco, A. (2022). The seasonal cycle will be filtered out.

Applications

- Climate dynamics

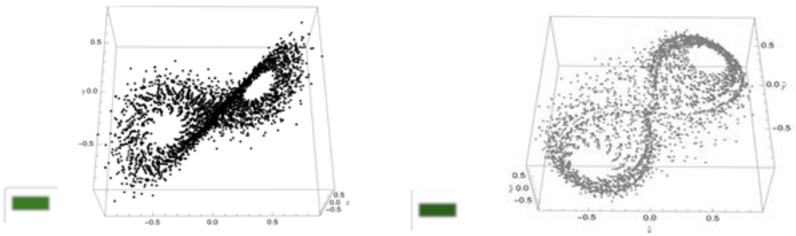
The dynamical properties of a climate attractor depend on its local and instantaneous properties, rather than its average properties [Lucarini et al, 2016]. Time series for the analysis will have a lower time resolution than that used in [Brunetti et al, 2019], filtering out the seasonal cycle.



Evolution of four attractors with sliding time windows of 1000 days (range 5000 days) and daily time resolution. The time series were calculated by Maura Brunetti specifically for Luciana Salvagni's graduate thesis.

Applications

- Climate dynamics



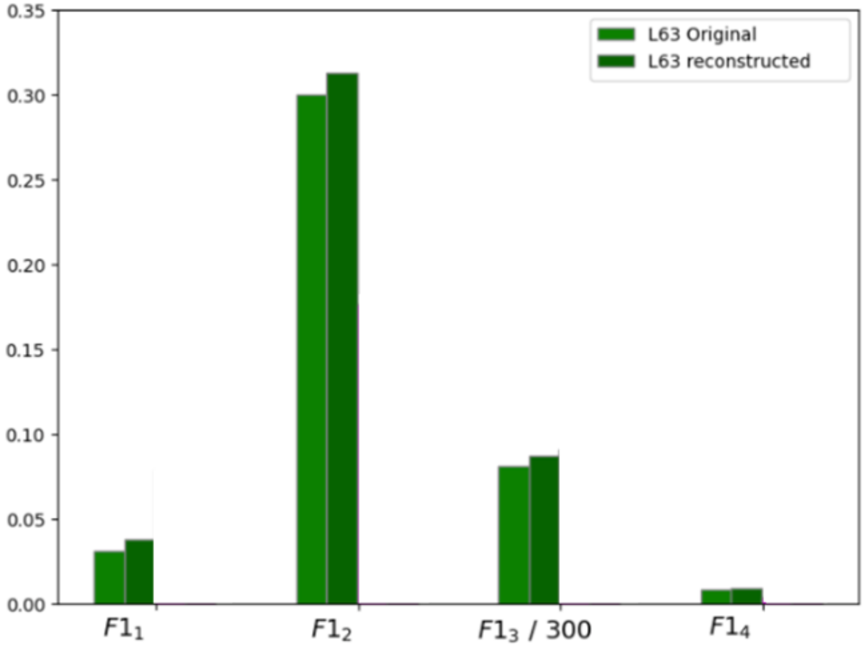
To analyze the persistence properties, topological markers can be defined to condense the salient features of holes.

$F1_1$: the start (birth) value of the largest 1-hole

$F1_2$: lifetime of the largest 1-hole (reflects the size of the geometrically dominant 1-hole)

$F1_3$: sum of the half-lives of all 1-holes

$F1_4$: averaged lifetime of the 1-holes (indicative of the average size of the 1-holes)



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Applications

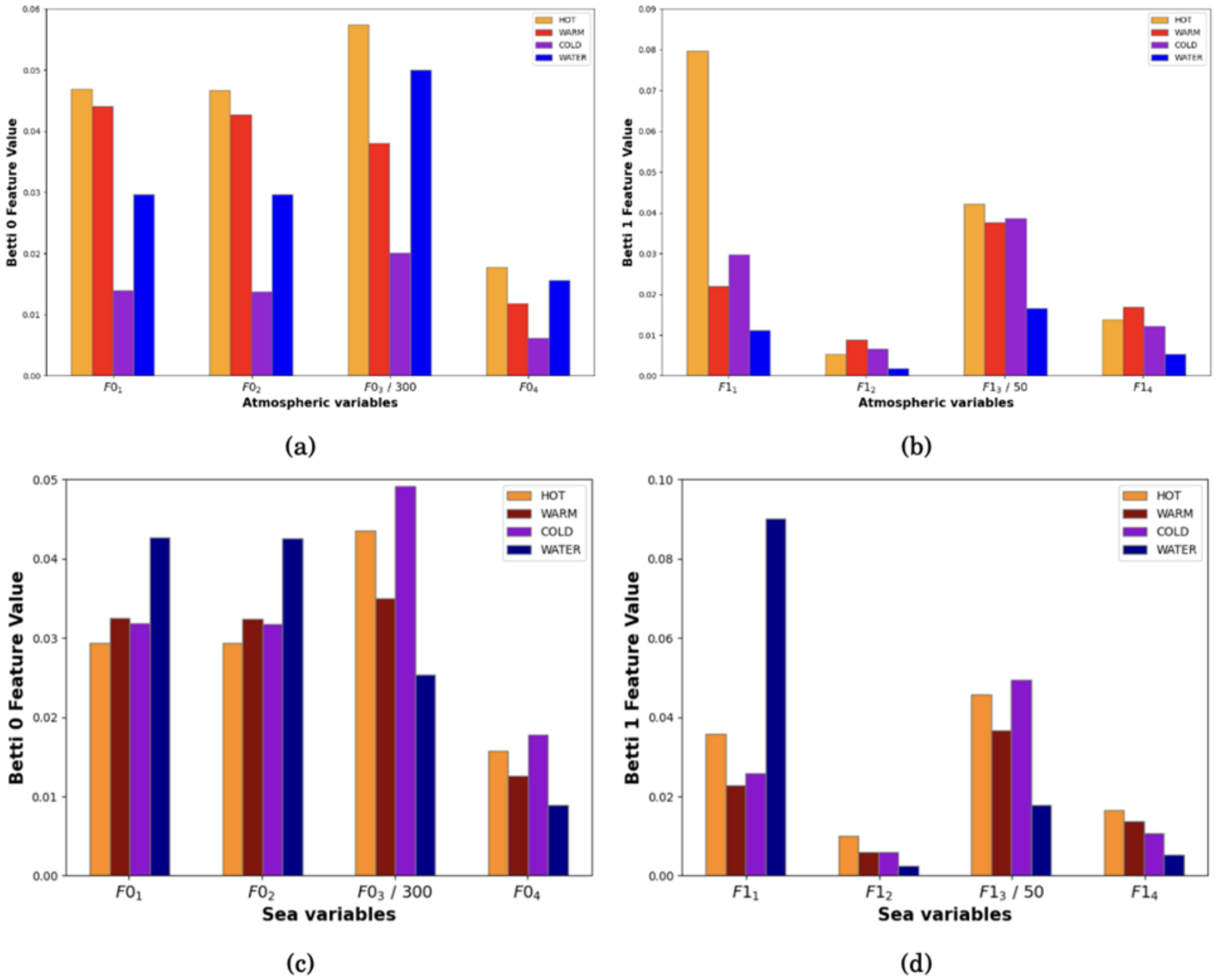
- Climate dynamics

Each attractor has a predominant trait in terms of persistent homologies that distinguishes it from the others in time windows of one thousand days.

The topological structure of each attractor is not yet unveiled: it cannot be condensed into a single representative cell complex.

Persistence diagrams cannot be used, at this stage, to obtain a BraMAH complex.

Topological markers for the four climatic attractors



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Templex

"There are more things in Topology and Dynamics, than are dreamt of in Homologies."

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Templex: A bridge between homologies and templates for chaotic attractors

Gisela D. Charó, Christophe Letellier and Denisse Sciamarella

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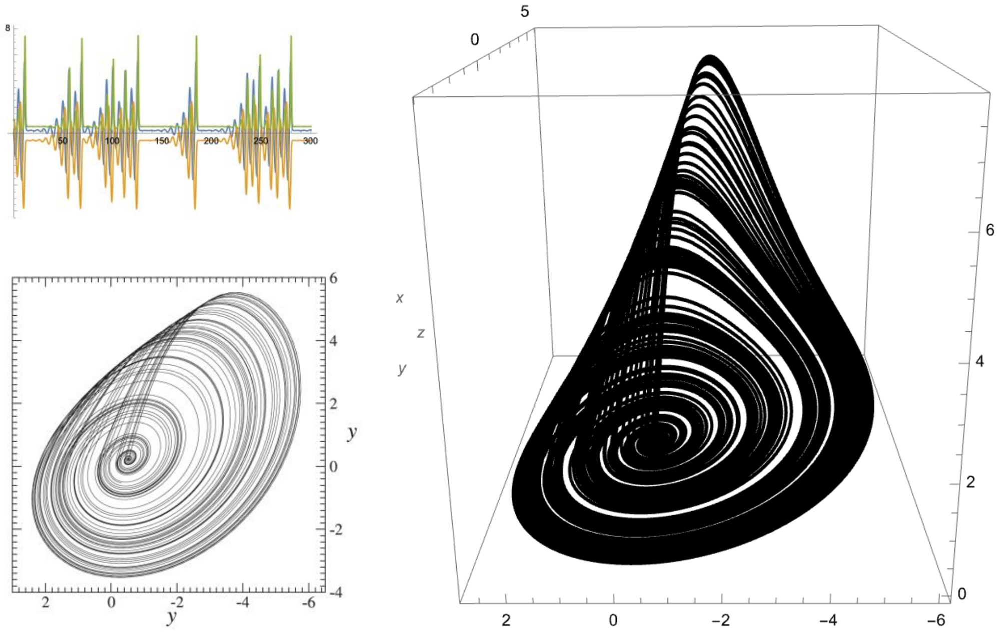
Templex

- Why and how was it conceived?

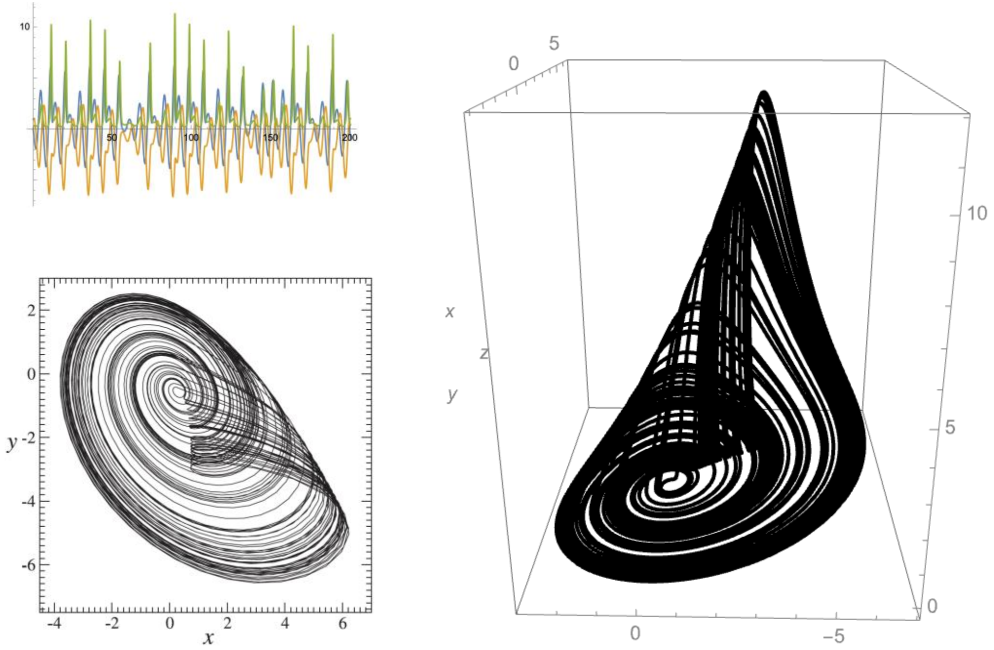
$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c). \end{cases}$$

Homologies cannot distinguish between two different attractors produced by the Rössler dynamical system with different parameter values (spiral Rössler attractor with a = 0.343295 on the left and the funnel Rössler attractor with a = 0.492 on the right).

Spiral Rössler attractor



Funnel Rössler attractor

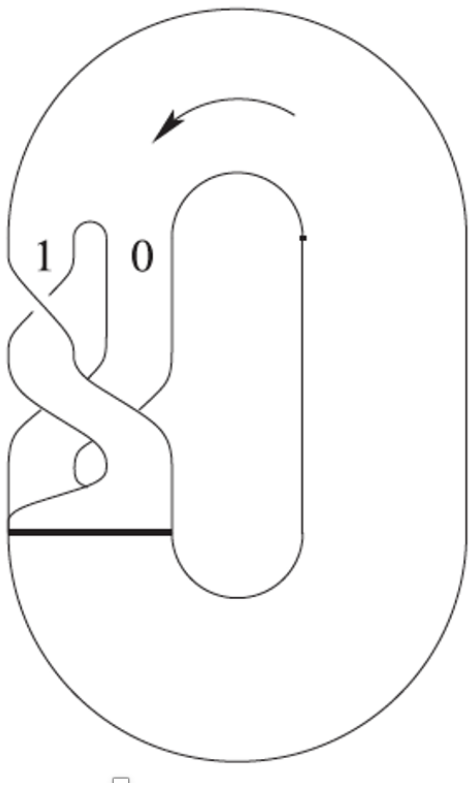


Templex

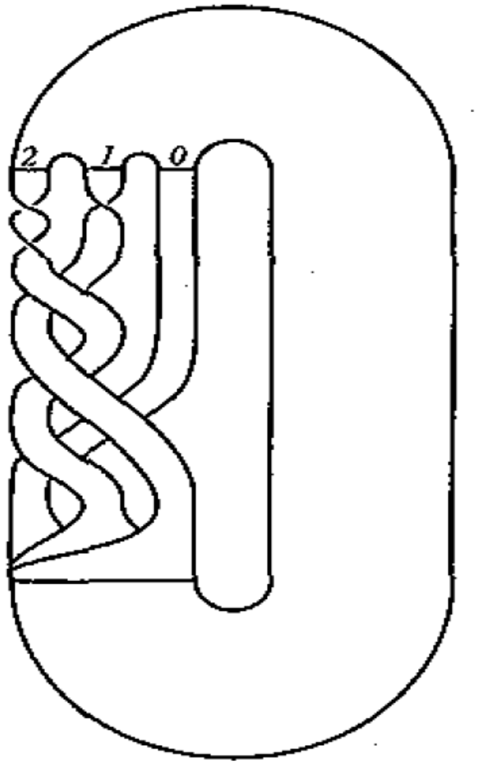
- Why and how was it conceived?

The template does distinguish between the two: the spiral Rössler attractor has two strips (0, 1), while the funnel Rössler attractor has three strips (0,1,2).

Spiral Rössler attractor



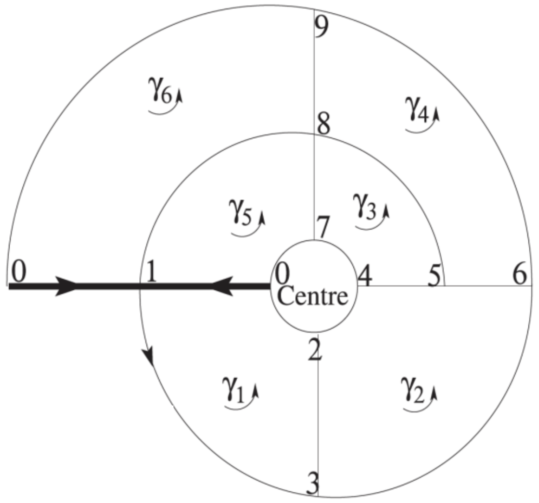
Funnel Rössler attractor

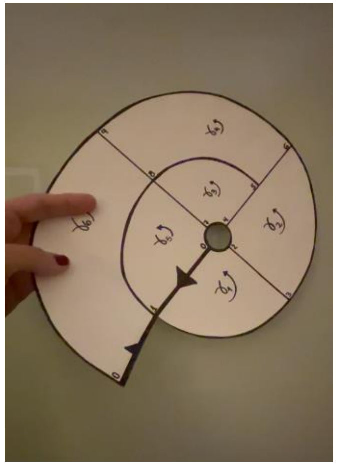


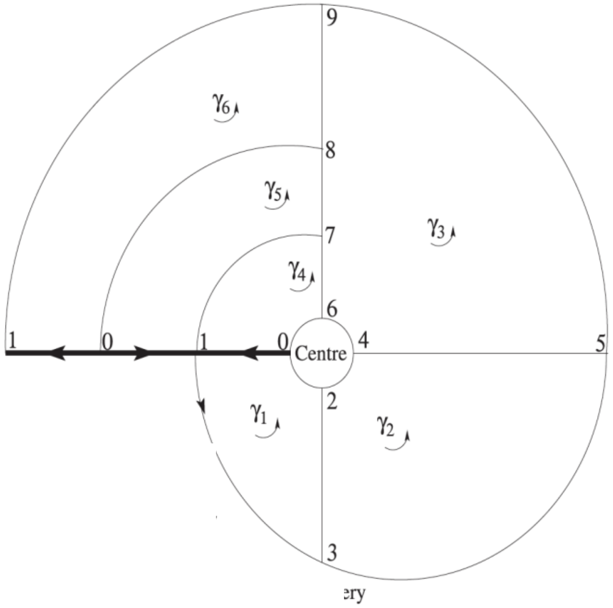
Templex

- Why and how was it conceived?

The spiral and funnel Rössler attractors are homologically equivalent: **they have both one hole in the centre** ($H_1 = Z^1$).







$$\mathcal{H}_1(K(\mathrm{R})) = [[\langle 0,2\rangle - \langle 0,7\rangle + \langle 2,4\rangle + \langle 4,7\rangle]]$$

$$\mathcal{H}_1(K(\mathrm{R}_3)) = [[\langle 0,2\rangle - \langle 0,6\rangle + \langle 2,4\rangle + \langle 4,6\rangle]]$$

But there is more information in a cell complex than the one contained in its homology groups… for instance, the joining lines! They can be detected as the 1-cells shared by at least three 2-cells (heavy lines).
Notice that the recipe to scotch the cell complexes is different.

Templex

- Why and how was it conceived?

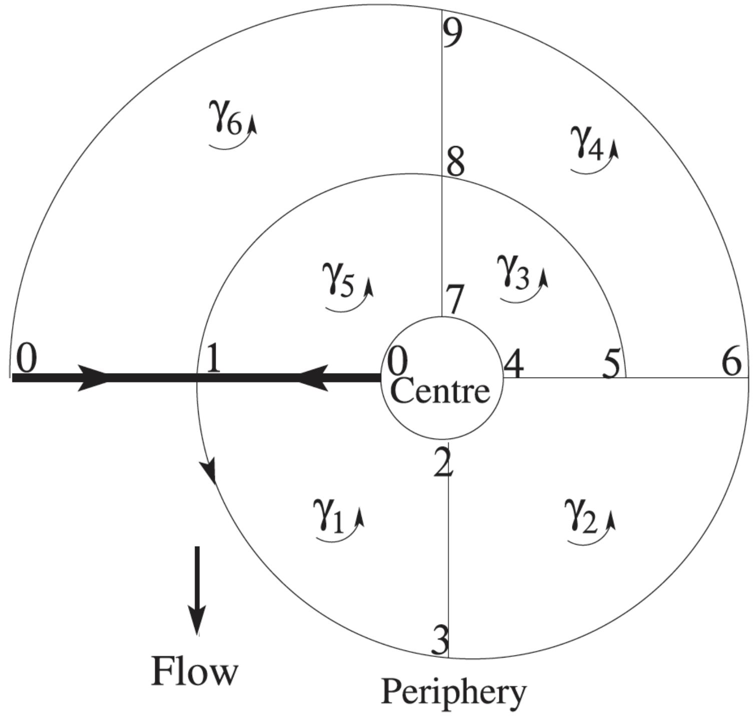
But there is something else that is very important and that is missing in a cell complex representing a branched manifold.

This is the *flow* or, in other words, *time*!

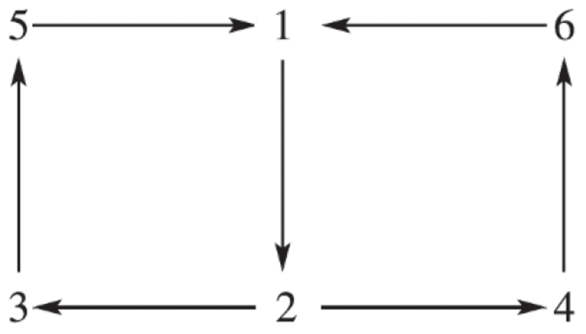
Homologies do not consider the action of the flow on the cell complex…

In order to take the flow on the complex into account, the cell complex will be endowed with a directed graph that prescribes the flow direction between its highest-dimensional cells.

Definition 1. A **templex** $T \equiv (K, G)$ is made of a complex K of dimension $\dim(K) = \kappa$ and a digraph $G = (N, E)$ whose underlying space is a branched κ-manifold associated with a dynamical system, such that (i) the nodes N are the κ-cells of K and (ii) the edges E are the connections between the κ-cells allowed by the flow.



(a)



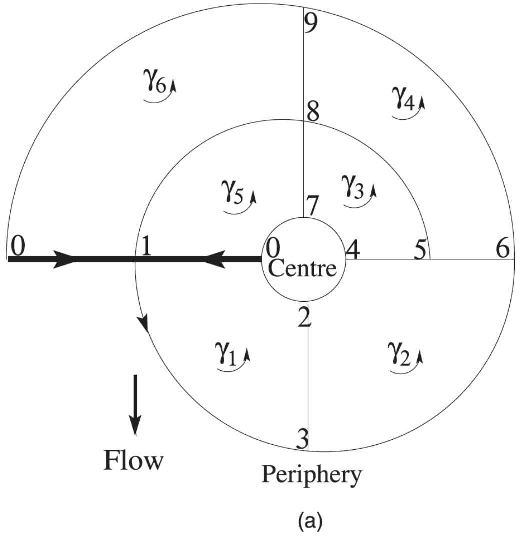
(b)

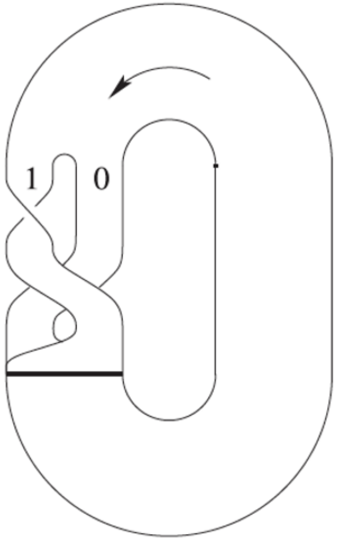
Templex

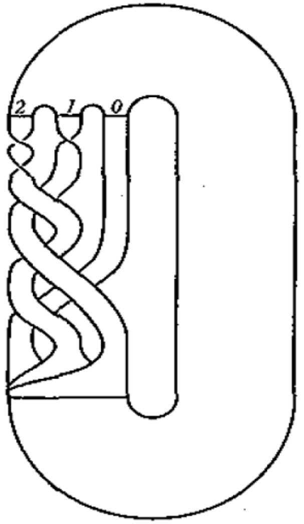
- Why and how was it conceived? How is it computed?

Spiral Rössler attractor

T(R)=(K(R),G(R))

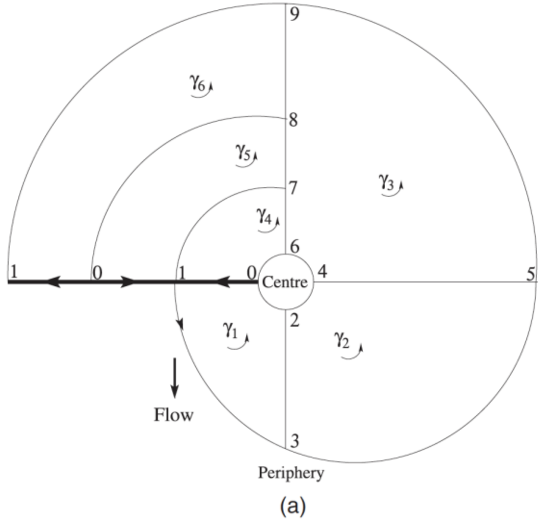






Funnel Rössler attractor

$T(R3)=(K(R_3),G(R_3))$



<u>2 Stripexes</u> in T(R):

twisted $\underline{1} \to 2 \to 4 \to 6 \to \underline{1},$

$\underline{1} \to 2 \to 3 \to 5 \to \underline{1}.$

<u>3 Stripexes</u> in $T(R_3)$:

$\underline{1} \to 2 \to 3 \to 4 \to \underline{1},$

twisted $\underline{1} \to 2 \to 3 \to 5 \to \underline{1},$

$\underline{1} \to 2 \to 3 \to 6 \to \underline{1},$

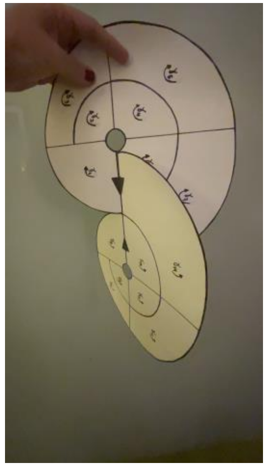
Templex

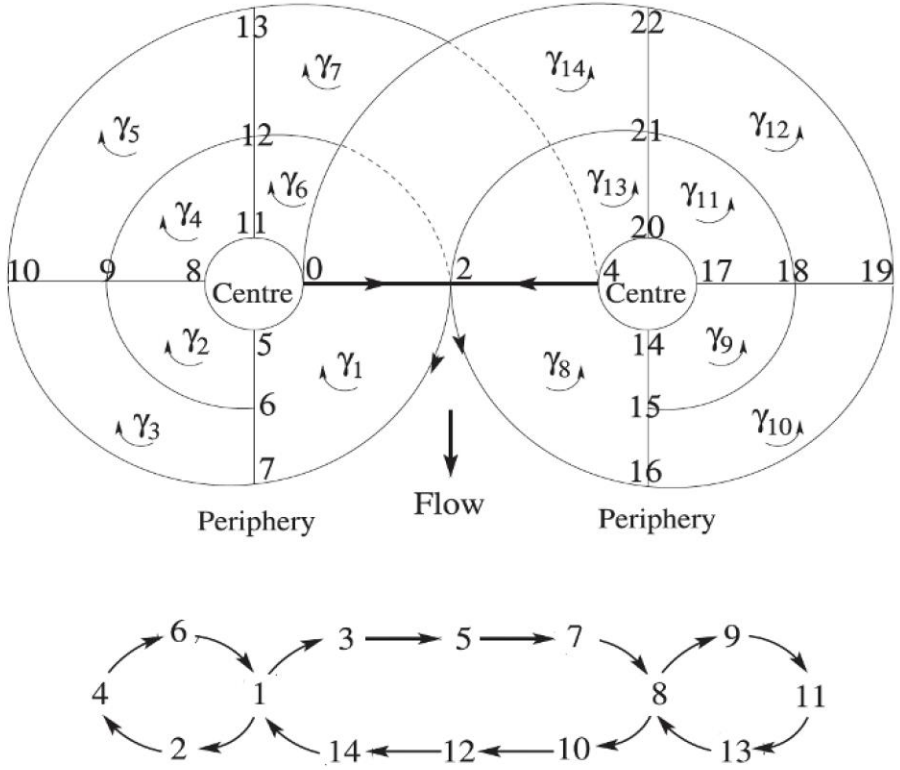
- Example I: Lorenz 63 example (autonomous)

T(L)=(K(L),G(L))

T'(L)=(K'(L),G'(L))

(with fewer cells)





Four stripexes in T(L):

$$c_1 \equiv \underline{1} \to 2 \to 4 \to 6 \to \underline{1},$$

$$c_2 \equiv \underline{8} \to 9 \to 11 \to 13 \to \underline{8},$$

strong cycles

$$c_{3_1} \equiv \underline{1} \to 3 \to 5 \to 7 \to \underline{8}$$

$$c_{3_2} \equiv \underline{8} \to 10 \to 12 \to 14 \to \underline{1}.$$

weak cycles

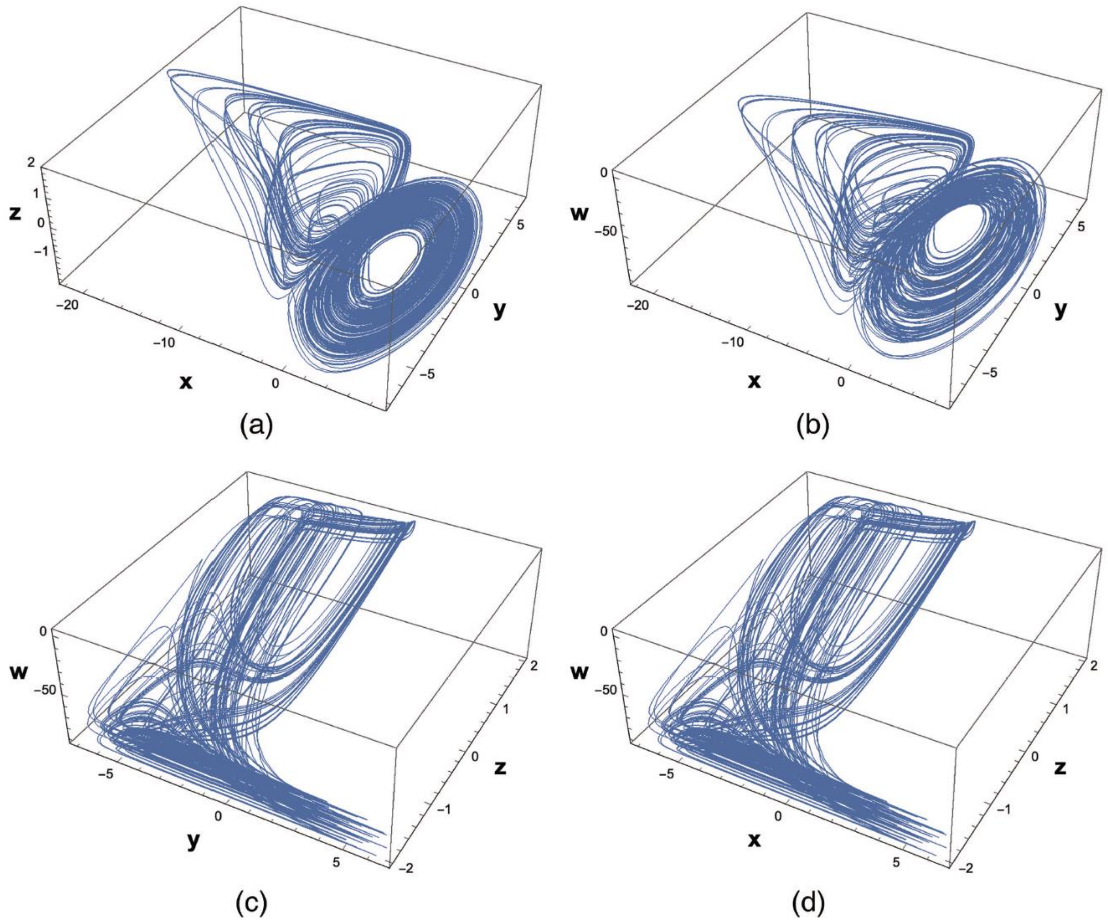
The weak cycles that form the **two twisted stripexes** correspond to a single generatex of order 2.

Templex

- Example II: 4D Deng attractor (autonomous)

A four-dimensional system designed from a three-dimensional system proposed by Deng.

$$\begin{cases} \dot{x} = -(z+2)d\,(x - [a + \epsilon_3(2+w)]) + (2-z) \\ \qquad \left[\alpha(x-2) - \beta y - \alpha\,(x-2)\,\dfrac{(x-2)^2 + y^2}{R^2}\right], \\ \dot{y} = -(z+2)\,(y-b) + (2-z) \\ \qquad \left[\beta\,(x-2) + \alpha y - \alpha y \dfrac{(x-2)^2 + y^2}{R^2}\right], \\ \dot{z} = (4 - z^2)\,\dfrac{z + 2 - \mu(x+2)}{\epsilon_1} - cz, \\ \dot{w} = (4 - z^2)\,\dfrac{z + 2 - \mu(x+2)}{\epsilon_2} - cz. \end{cases}$$



A solution to this system was already investigated with a BraMAH cell complex (but not with a templex) in Sciamarella & Mindlin, 2001.

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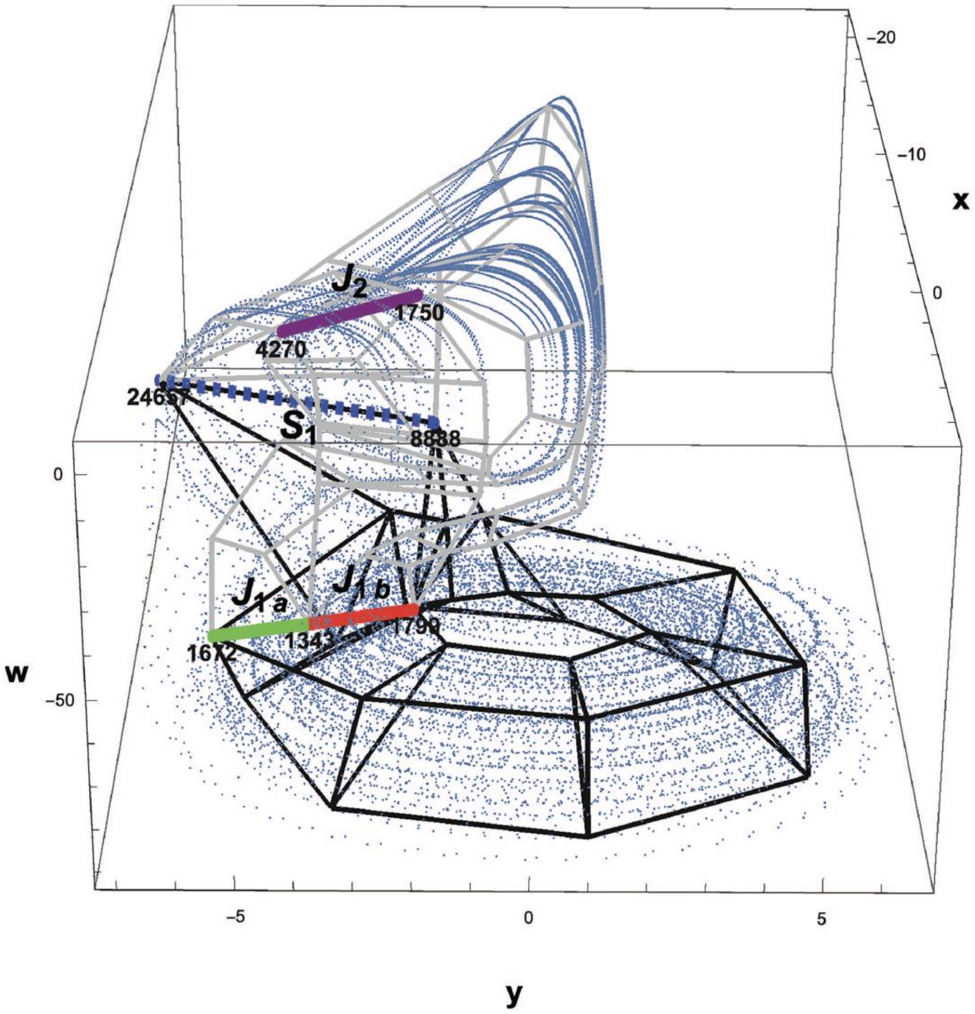
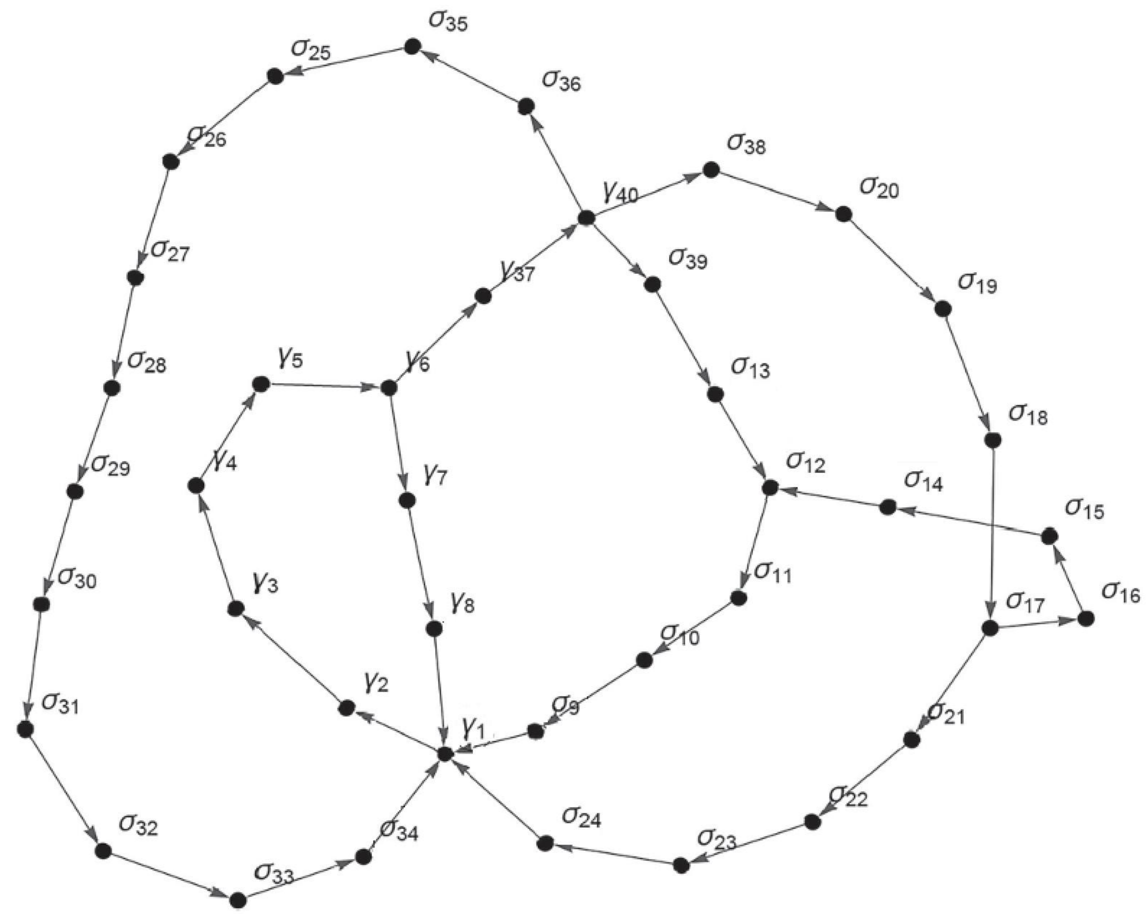
Templex

- Example II: 4D Deng attractor (autonomous)



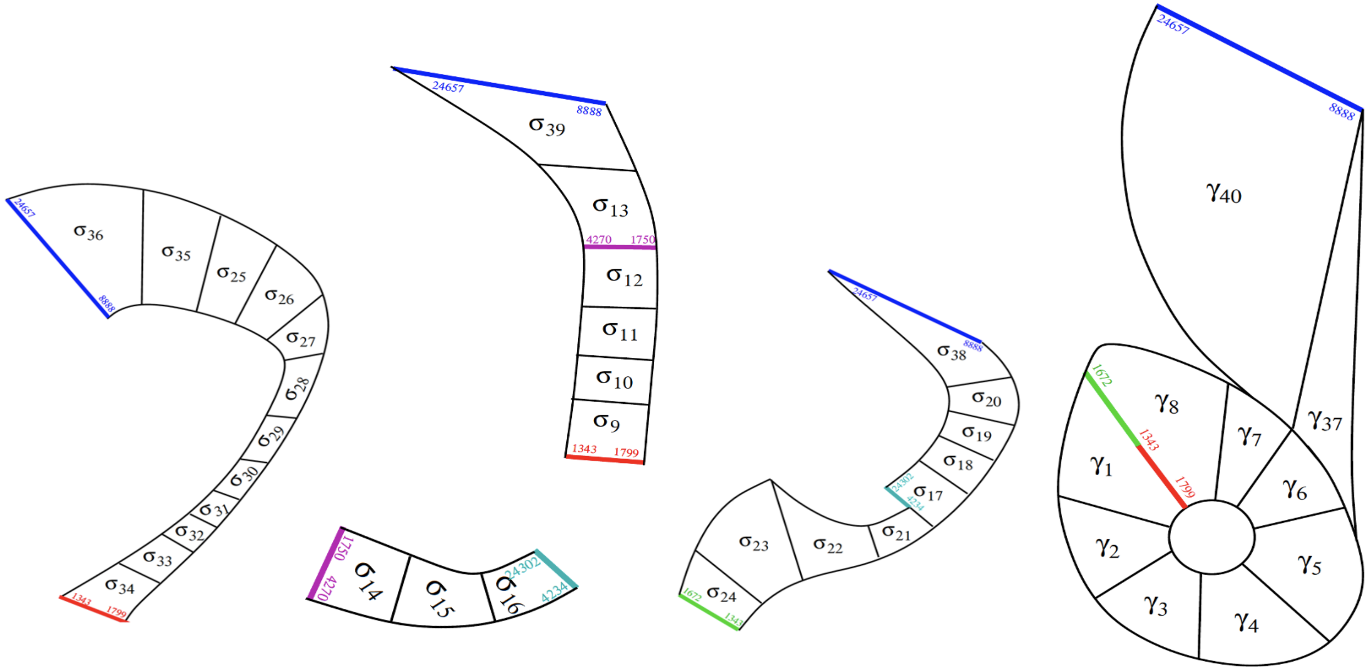
Five stripexes in T(4D)=(K(4D),G(4D))

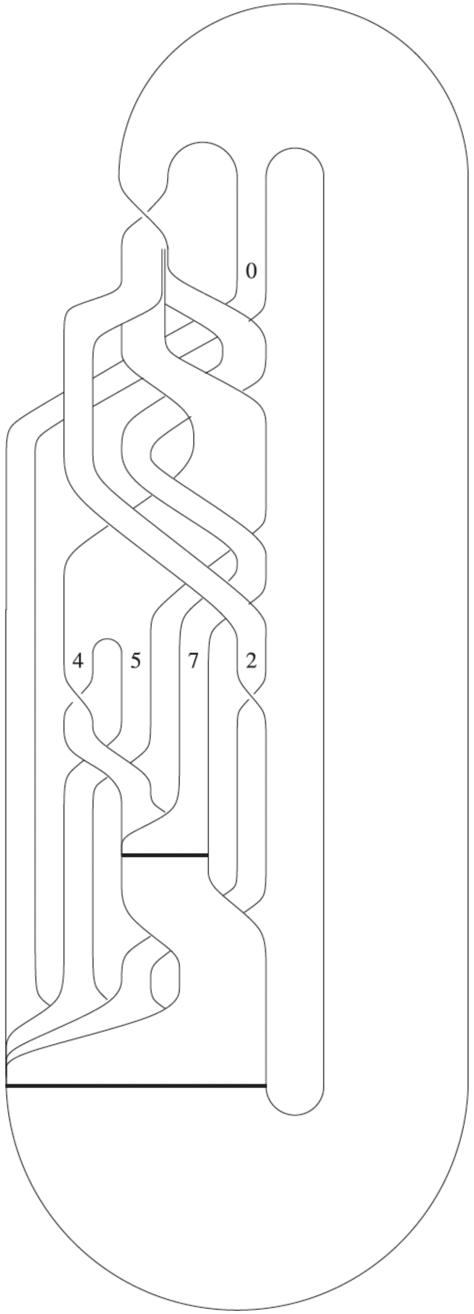


Templex

- Example II: 4D Deng attractor (autonomous)

The templex can be seen as dissecting the phase-space structure into several identifiable components, connected at certain joints with non-redundant pathways (stripexes) on it.



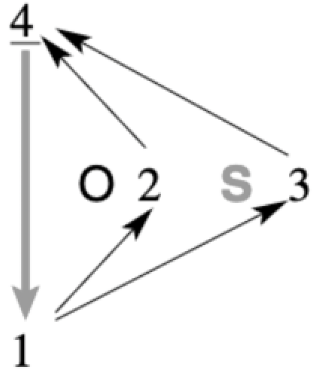


The templex properties:
- some describe the structure alone (holes, torsions)
- others describe the flow along the structure (stripex, twists).

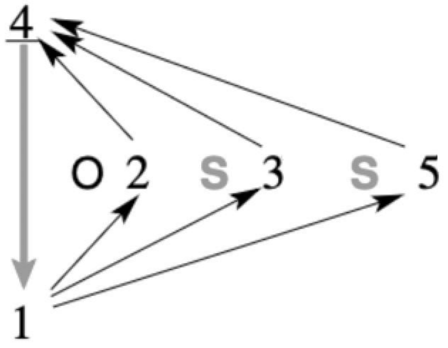
Templex

- Ongoing work:

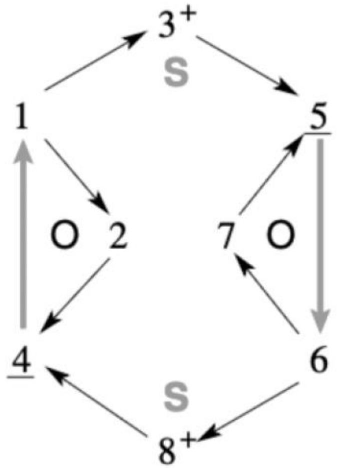
A reduced templex can be obtained using a set of topological rules. Cells in the BraMAH complex are merged if they do not add new information, and the digraph is redrawn in terms of main splitting and joining nodes.



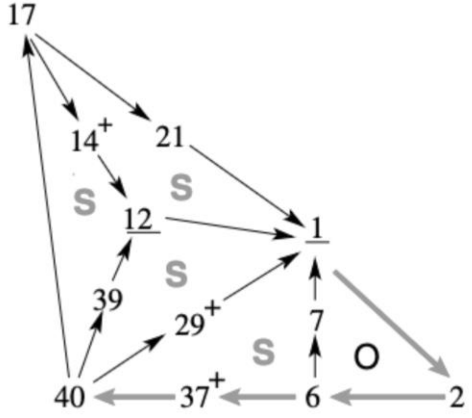
Spiral Rössler attractor



Funnel Rössler attractor



Lorenz attractor



4D Deng attractor

The reduction leads to a <u>combinatorial approach</u>. A certain type of dynamics is obtained by assembling <u>fundamental dynamical units</u> of two types: O and S.

Caterina Mosto, Gisela D. Charó, C. Letellier & Denisse Sciamarella, Templex-based dynamical units for a taxonomy of chaos, in preparation.

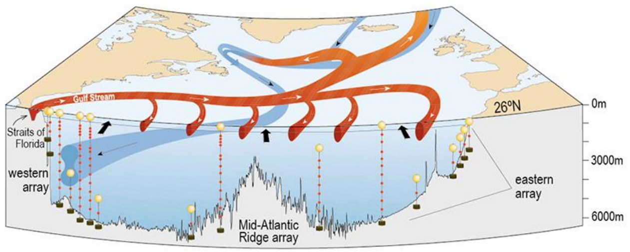
Templex

Work in progress with C. Mosto, G. Charó & J. Ruiz in collaboration with F. Sévellec (LOPS, Brest).

- Example III: 3D AMOC example (autonomous)

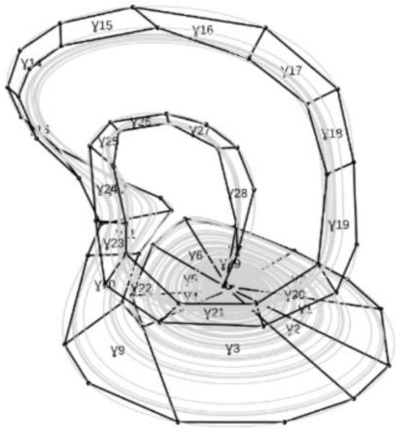
Let us now consider an autonomous 3D model of the Atlantic Meridional Overturning Circulation (AMOC) in Sévellec et Fedorov (*J. Clim.*, 2014) reproducing the chaotic dynamics of the Quaternary glaciations.

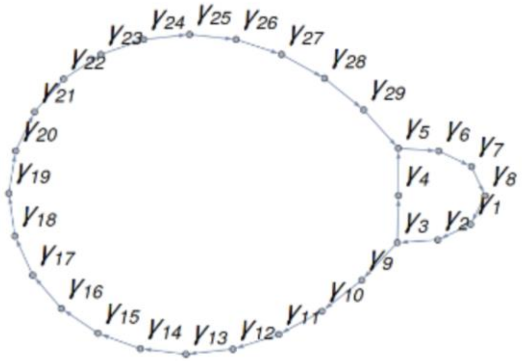
$$d_t\omega = -\lambda\omega - \epsilon\beta S_{NS},$$
$$d_t S_{BT} = +\Omega S_{NS} - K S_{BT} + F_{BT},$$
$$d_t S_{NS} = -\Omega S_{BT} - K S_{NS} + F_{NS}.$$



The simplest AMOC model has the same set of stripexes as the spiral Rössler attractor.

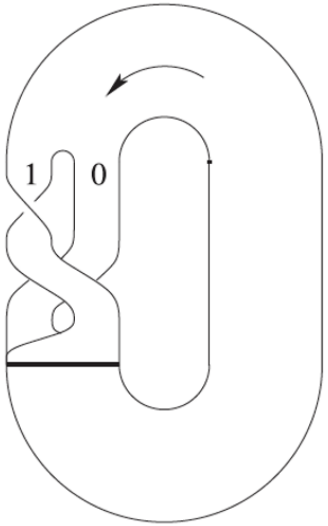
T(AMOC3D) = (K(AMOC3D), G(AMOC3D))





K(AMOC3D) has two 1-holes

T(AMOC3D) has two stripexes, one of which is twisted.



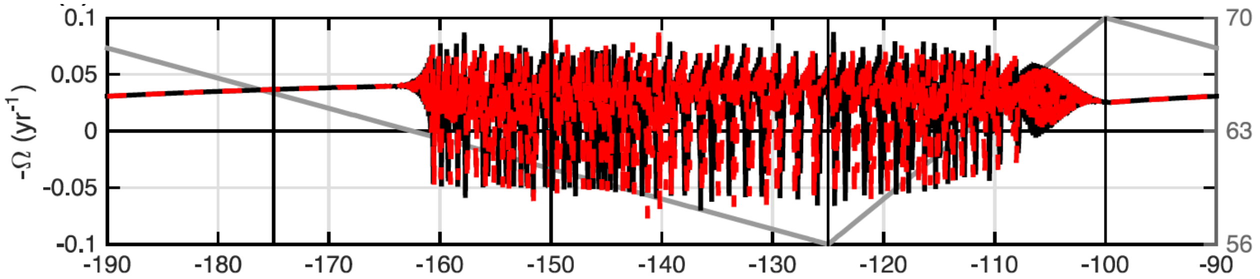
Templex

Work in progress with C. Mosto, G. Charó & J. Ruiz in collaboration with F. Sévellec (LOPS, Brest).

- Example IV: Unstable AMOC (non autonomous)

Let us consider the imposed temporal changes in the position of the edge of sea ice (ESI) to account for the chaotic behavior during the glacials and for the stable ocean conditions during the interglacials.. F_{BT} and F_{NS} are the Fourier projections of surface salt flux.

$$d_t\omega = -\lambda\omega - \epsilon\beta S_{NS},$$
$$d_t S_{BT} = +\Omega S_{NS} - K S_{BT} + F_{BT},$$
$$d_t S_{NS} = -\Omega S_{BT} - K S_{NS} + F_{NS},$$
$$\text{where } F_{BT} + iF_{NS} = \frac{S_0}{2\pi h}\int_0^{2\pi} Fe^{-i\theta}d\theta.$$



Simulated variations in the overturning rate (−Ω) for two slightly different sets of initial conditions (solid black and dashed red lines) for a single glacial–interglacial cycle.

The grey sawtooth line indicates the imposed temporal changes in the position of the edge of sea ice (ESI).

The four vertical lines indicate the freezing times (t_0) used later to compute the Pullback attractor (**PBA**) [Ghil et al, 2008; Chekroun et al, 2011].

Templex

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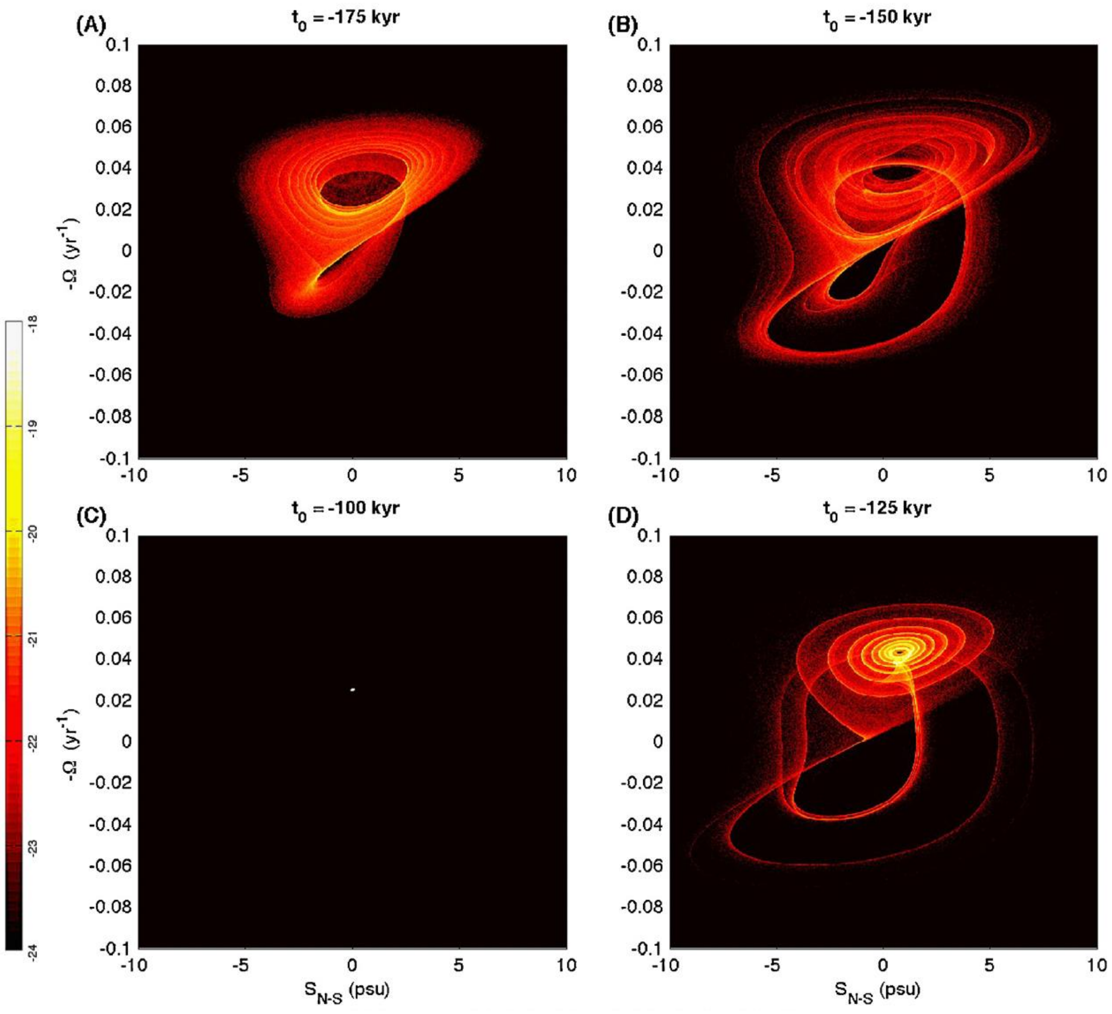
- Example IV: Unstable AMOC (non autonomous)



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Unstable AMOC during glacial intervals and millennial variability: The role of mean sea ice extent

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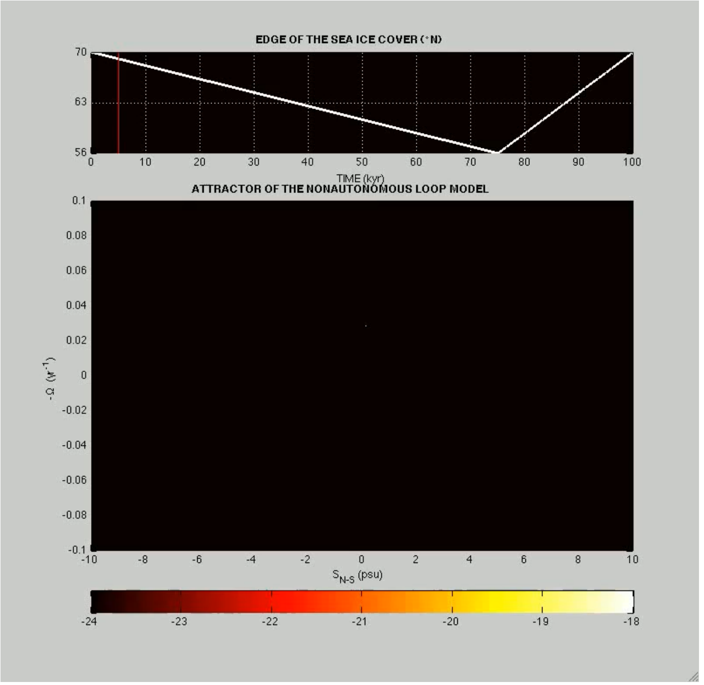


Metodi matematici per la scienza del clima

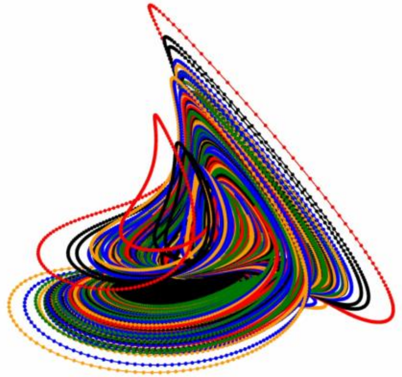
Work in progress with C. Mosto, G. Charó & J. Ruiz in collaboration with F. Sévellec (LOPS, Brest).

Templex

- Example IV: Unstable AMOC (non autonomous)

4D solutions: we obtain a cloud of four-dimensional points. We construct the **BraMAH** complex from this point cloud. The four-dimensional point cloud does not have false neighbors: it is related to an autonomous writing of the AMOC equations and can be used to build a templex.

-1.204390949016654645e-02	-3.755666436397816499e-01	1.306518394821738482e+00	1.348398174857631283e-02
-1.224216393305090946e-02	-4.105094258159222020e-01	1.302943763424306400e+00	1.348421537371332669e-02
-1.243740567030636977e-02	-4.455459503156868895e-01	1.297985263600033523e+00	1.348444899293549729e-02
-1.262932547193569219e-02	-4.806182011675594890e-01	1.291628288655764401e+00	1.348468260624273440e-02
-1.281761394464031763e-02	-5.156665152828940890e-01	1.283860898299330700e+00	1.348491621363492356e-02
-1.300196222019738522e-02	-5.506297133049842252e-01	1.274673984047107300e+00	1.348514981511196587e-02
-1.318206265102916247e-02	-5.854452473115573374e-01	1.264061403415025930e+00	1.348538341067376072e-02
-1.335760953344140682e-02	-6.200493585640195482e-01	1.252020112163938492e+00	1.348561700032020402e-02
-1.352829982505172543e-02	-6.543772419896768389e-01	1.238550270411638454e+00	1.348585058405119691e-02
-1.369383396027380913e-02	-6.883632413582051468e-01	1.223655375731521167e+00	1.348608416186663009e-02
-1.385391662238839167e-02	-7.219410390573020031e-01	1.207342364900948040e+00	1.348631773376640641e-02
-1.400825754992730994e-02	-7.550438634653665604e-01	1.189621705883329605e+00	1.348655129975042180e-02
-1.415657236815926791e-02	-7.876047132207103507e-01	1.170507470205472522e+00	1.348678485981857564e-02
-1.429858339870929487e-02	-8.195565760147025536e-01	1.150017391270899081e+00	1.348701841397076037e-02
-1.443402050473086007e-02	-8.508326700754310634e-01	1.128172909629967080e+00	1.348725196220687365e-02
-1.456262192774995280e-02	-8.813666883084957382e-01	1.104999200382378088e+00	1.348748550452682354e-02
-1.468413509473222647e-02	-9.110930350798221999e-01	1.080525186846978958e+00	1.348771904093049555e-02
-1.479831746114401020e-02	-9.399470891449953625e-01	1.054783525091836704e+00	1.348795257141779599e-02
-1.490493731082714705e-02	-9.678654534644385299e-01	1.027810576471543280e+00	1.348818609598861386e-02
-1.500377453792354869e-02	-9.947862054894188732e-01	9.996463597398603795e-01	1.348841961464285373e-02
-1.509462142964136840e-02	-1.020649159083170110e+00	9.703344758810172888e-01	1.348865312738041326e-02
-1.517728339478972496e-02	-1.045396106763873956e+00	9.399220267897770986e-01	1.348888663420118143e-02
-1.525157967480668918e-02	-1.068971066939994996e+00	9.084595025506267962e-01	1.348912013510506283e-02
-1.531734402118566096e-02	-1.091320526419735426e+00	8.760006462164051655e-01	1.348935363009195684e-02
-1.537442532355817737e-02	-1.112393665222866090e+00	8.426023160366005182e-01	1.348958711916175765e-02
-1.542268819799532883e-02	-1.132142579929327209e+00	8.083243076167658803e-01	1.348982060231436464e-02



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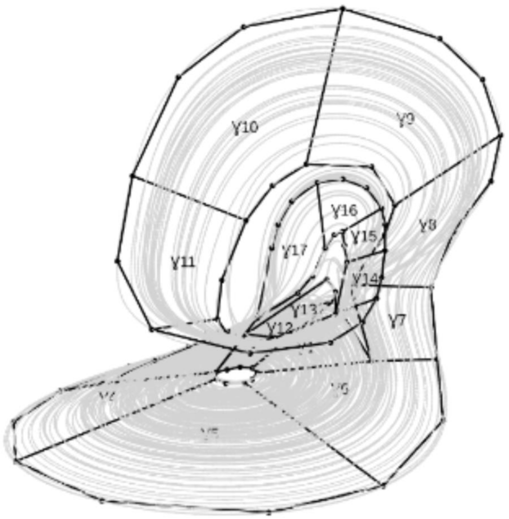
Templex

- Example IV: Unstable AMOC (non autonomous)

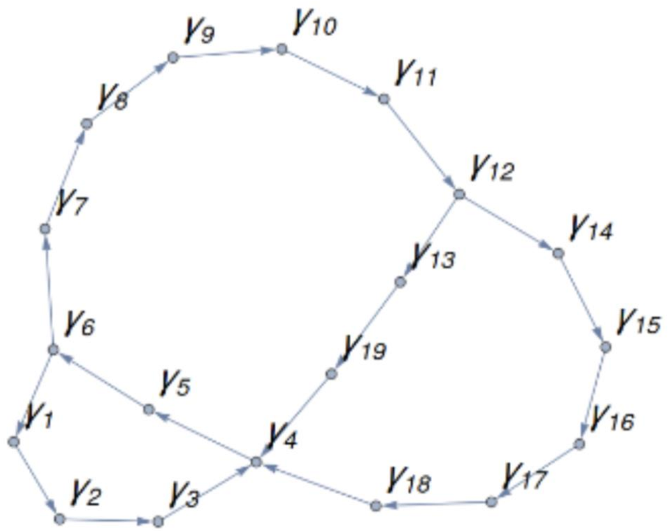
T(AMOC4D) = (K(AMOC4D), G(AMOC4D))



K(AMOC4D) has 3-cells (basis: solid torus)

Three 1-holes in the BraMAH complex

3 stripexes (1 twisted)



$$c_1 \equiv \underline{4} \to 5 \to 6 \to 1 \to 2 \to \underline{3}$$

$$c_2 \equiv 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11 \to 12 \to 13 \to 19$$

twisted $c_3 \equiv \underline{4} \to 5 \to 6 \to 7 \to 8 \to 9 \to 10 \to 11 \to 12 \to 14 \to 15 \to 16 \to 17 \to \underline{18}$

What is the relationship between the PBA approach and the 4D Templex structure?

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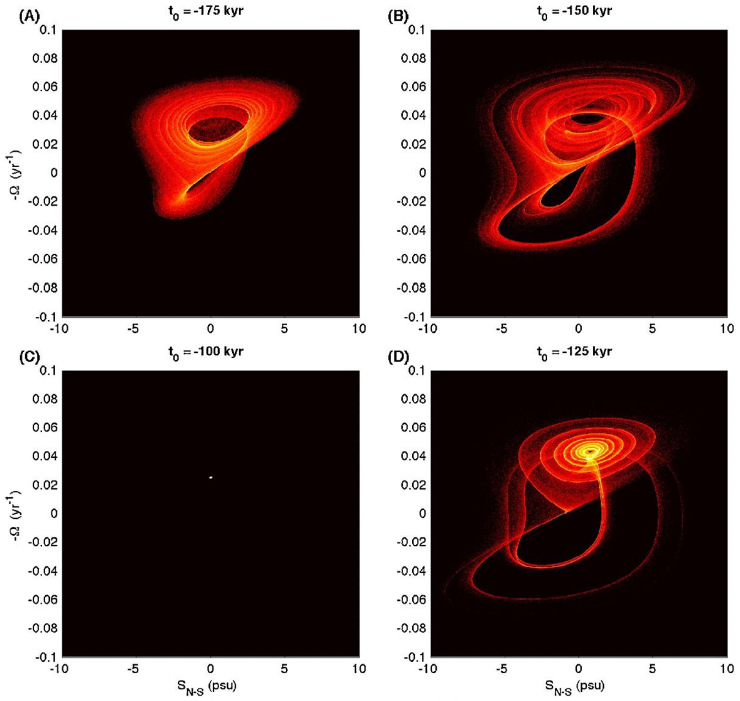
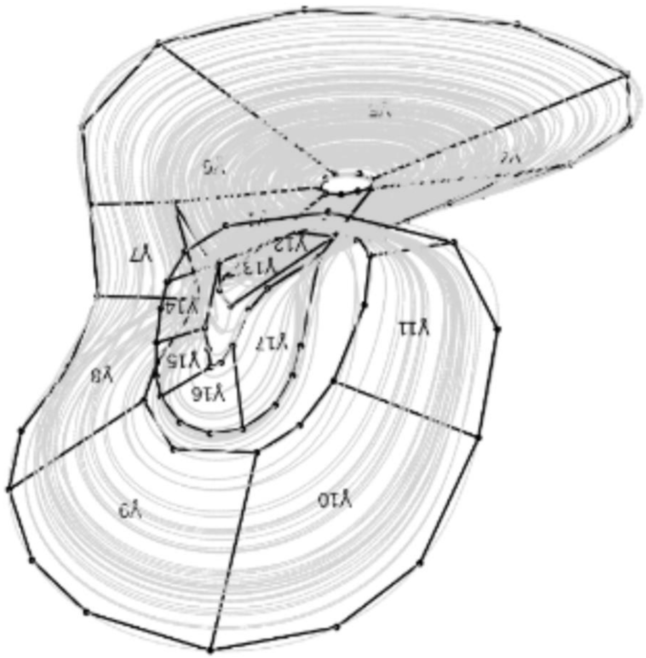
Templex

- Example IV: Unstable AMOC (non autonomous)

The **templex** can be considered as a single static object of higher dimension, combining all the « parts » of the structure observed in the snapshot sequence in a PBA approach as a fonction of (t_0,t).





Working in higher dimensions may provide an alternative to working with the PBA approach.

Università di Pisa

Metodi matematici per la scienza del clima

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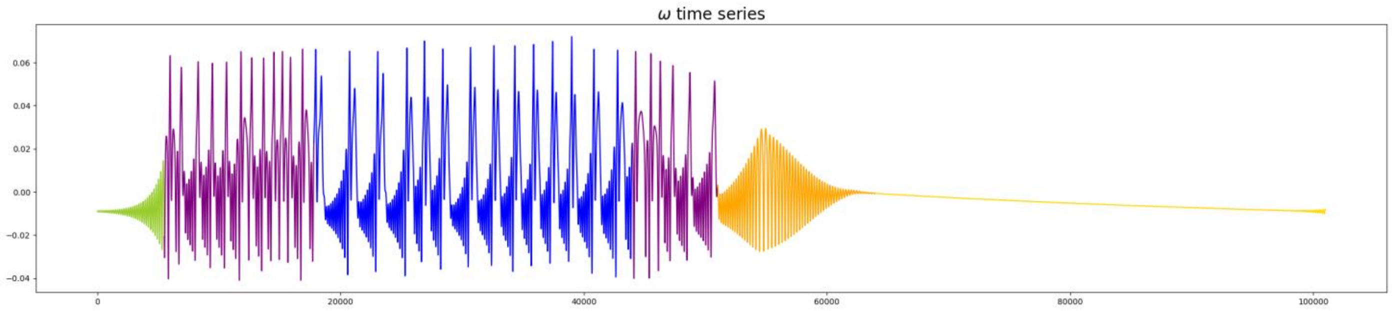
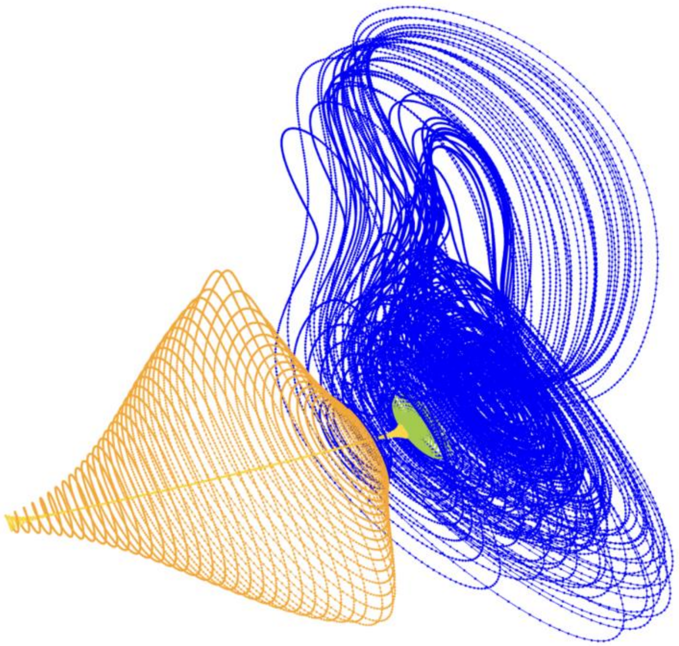
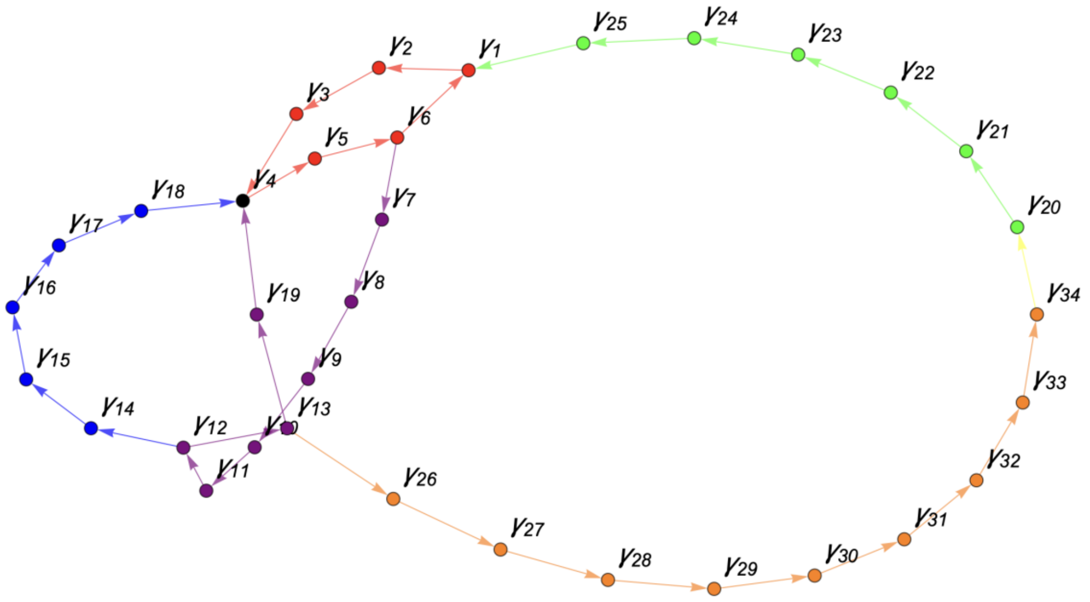
Templex

- Example IV: Unstable AMOC (non autonomous) with interglacial phase



Two 1-holes in the BraMAH complex: one of the 1-holes of the glacial phase is now covered by the orange/yellow/green phases.





A fourth stripex appears, going through the interglacial part of the templex.

The change of colors in the time series are related to the stripexes being visited.

Templex

- Computing templex from data

Developing an algorithm to compute the templex from an embedded time series.

Defense: February 27th, 2024

SECCIÓN 7. ARQUITECTURA DE LA SOLUCIÓN 56

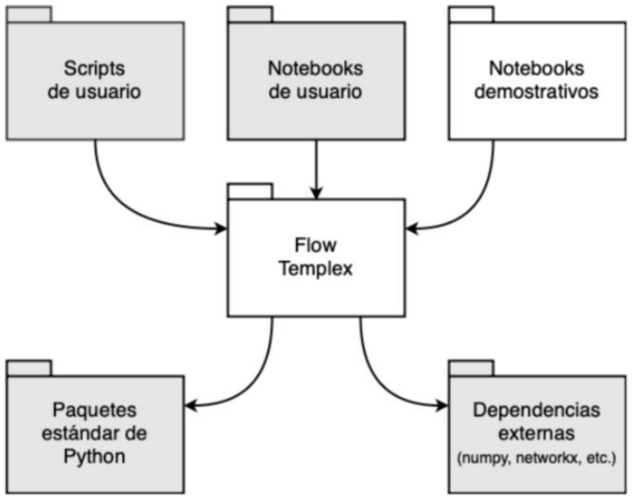


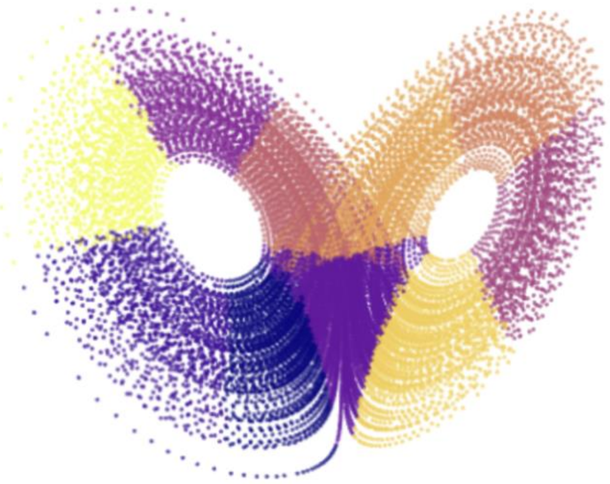
Figura 7.1: *Diagrama de paquetes que describe el contexto en el que el código del paquete "Flow Templex" existe. Los componentes con fondo blanco son los que fueron desarrollados para este trabajo, y los componentes con fondo gris son elementos externos. Las flechas indican dependencias entre los módulos.*

.UBAfiuba FACULTAD DE INGENIERÍA

Tesis de Ingeniería en Informática

Desarrollo de algoritmos para el estudio topológico de flujos caóticos

Adrián Matías Barreal



Random Templex

- How is it defined?

The templex was originally defined to describe the flow on a static branched manifold in phase space.

[Chekroun et al, 2011] showed that adding multiplicative noise can cause the structure in phase space to move.

The starting point is now a moving point cloud…

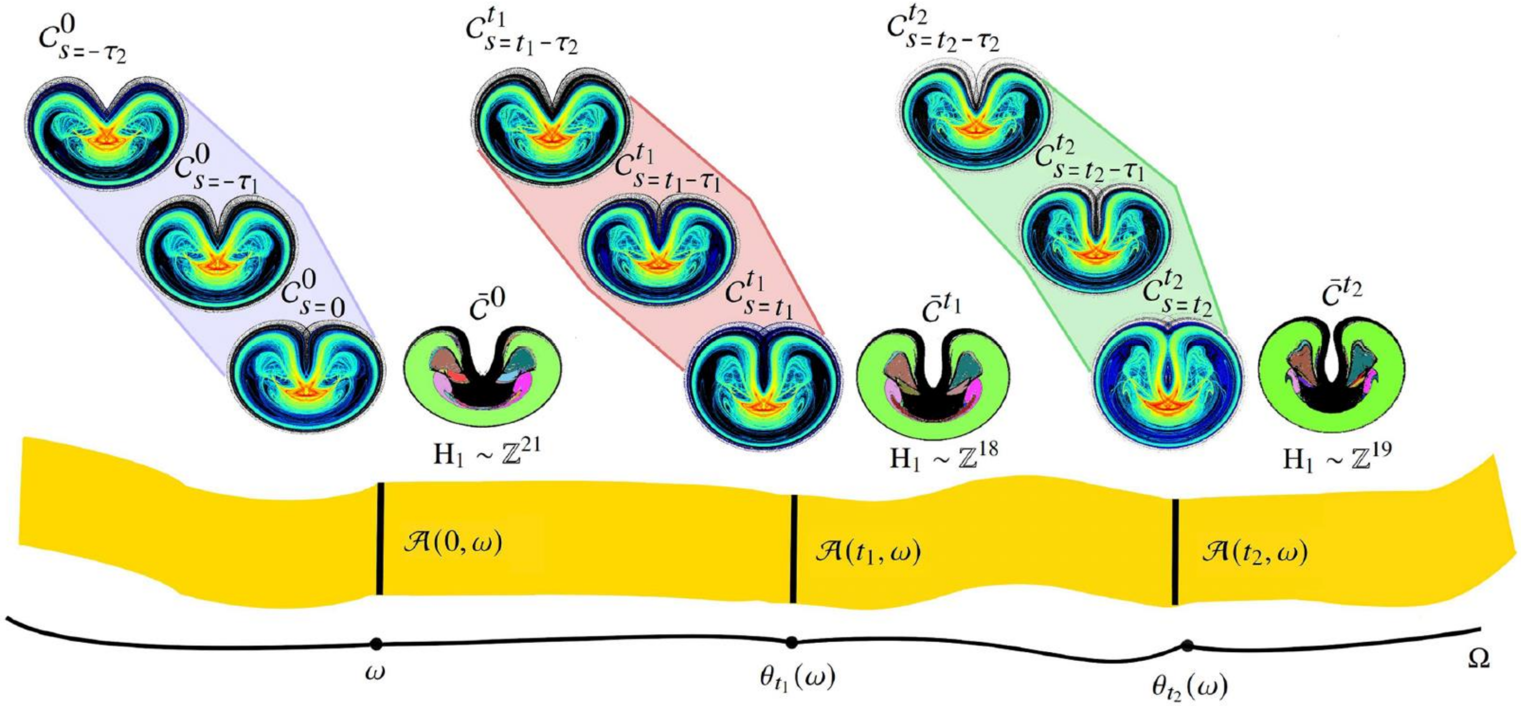
Computing topological invariants for **random attractors**

moving *n*-D point-cloud → Random templex

Random Templex

- How is it defined?

We now have cell complex per snapshot and homologies do not necessarily stay the same all the time. In the sequence below we show three snapshots, each with a different number of holes.

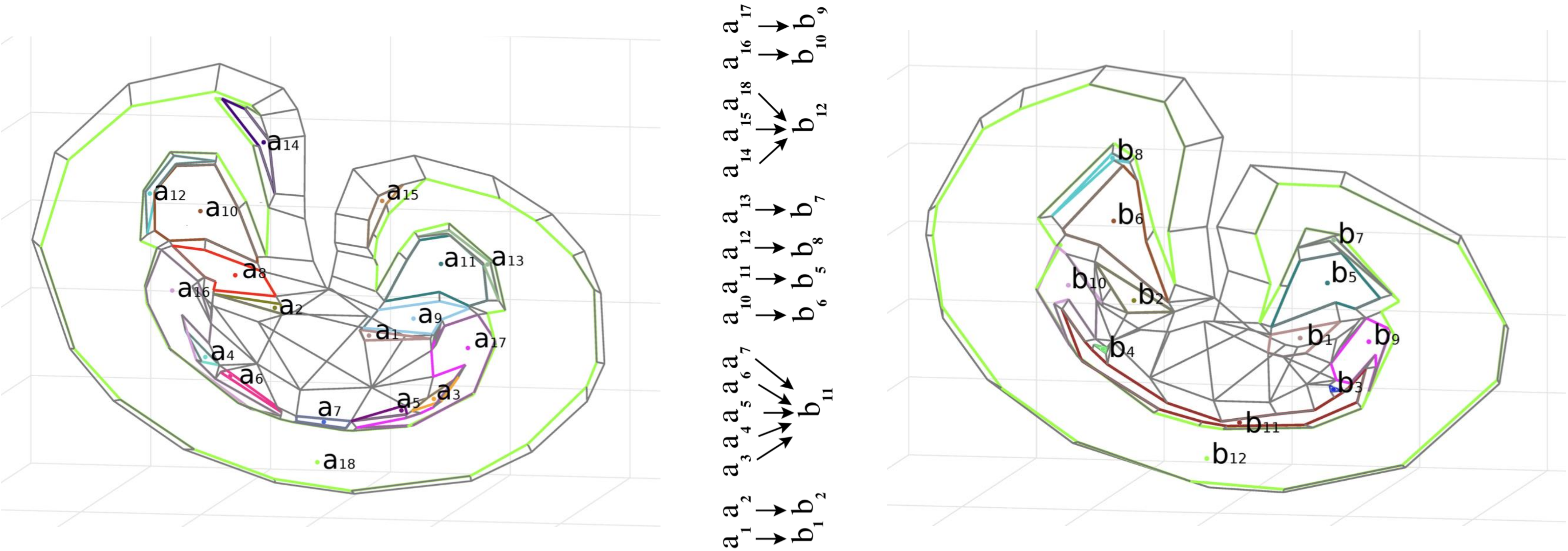


G. D. Charó, M. Ghil and D. Sciamarella: Random templex encodes topological tipping points in noise-driven chaotic dynamics. Chaos: An Interdisciplinary Journal of Nonlinear Science, 33 (10), pp.103141 (2023)

Random Templex

- How is it defined?

How can we track changes between different cell complexes? Tracking holes!



For a random attractor, the digraph does not connect cells, but holes of successive time cell complexes.

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Metodi matematici per la scienza del clima

Random Templex

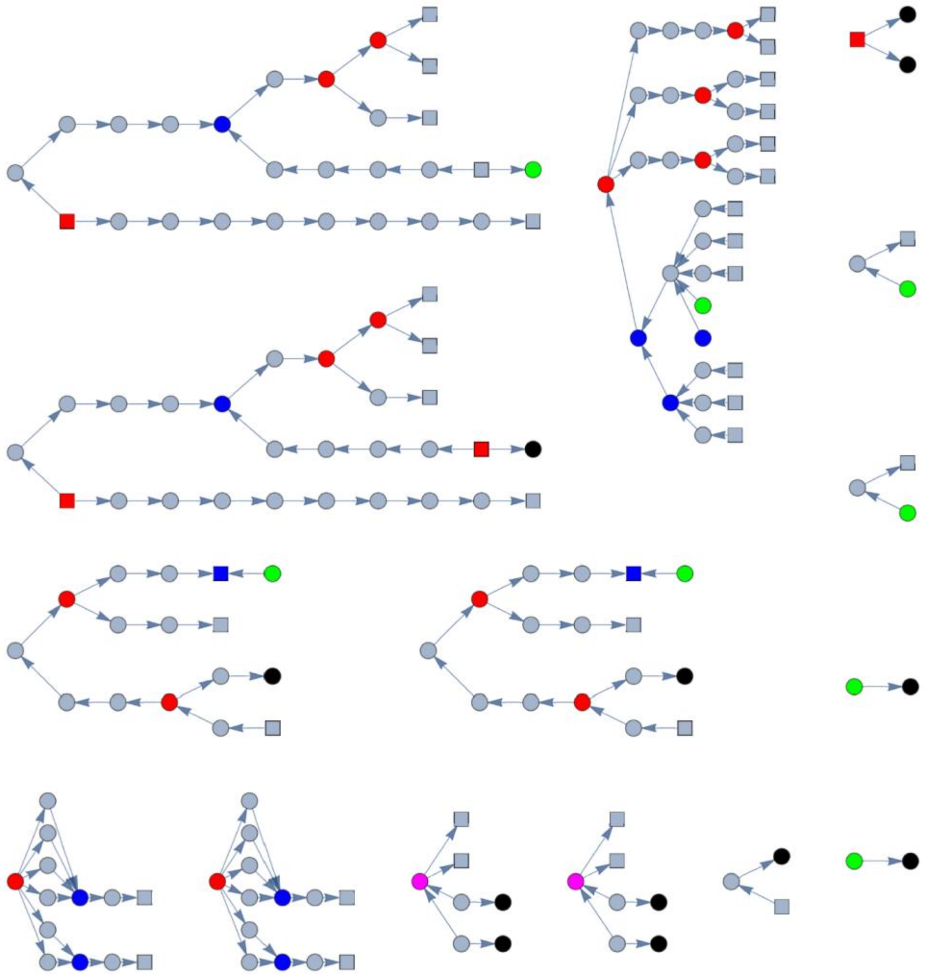
- How is it defined?

A random 2-templex $\mathcal{R} = (\mathcal{K}\text{‹}\ \mathcal{D})$ *is an indexed family* $\mathcal{K}$ *of BraMAH 2-complexes and a digraph* $\mathcal{D}$*.*

The digraph for LORA can be presented as a tree plot. It has 15 singly connected components, each of which tells the story of one or several holes.

Tipping points can be identified and classified using $\sqcup\langle\rceil\ \lceil\rangle\}\nabla\dashv\sqrt{\ }\langle$. They are highlighted in different colours according to the type of event:

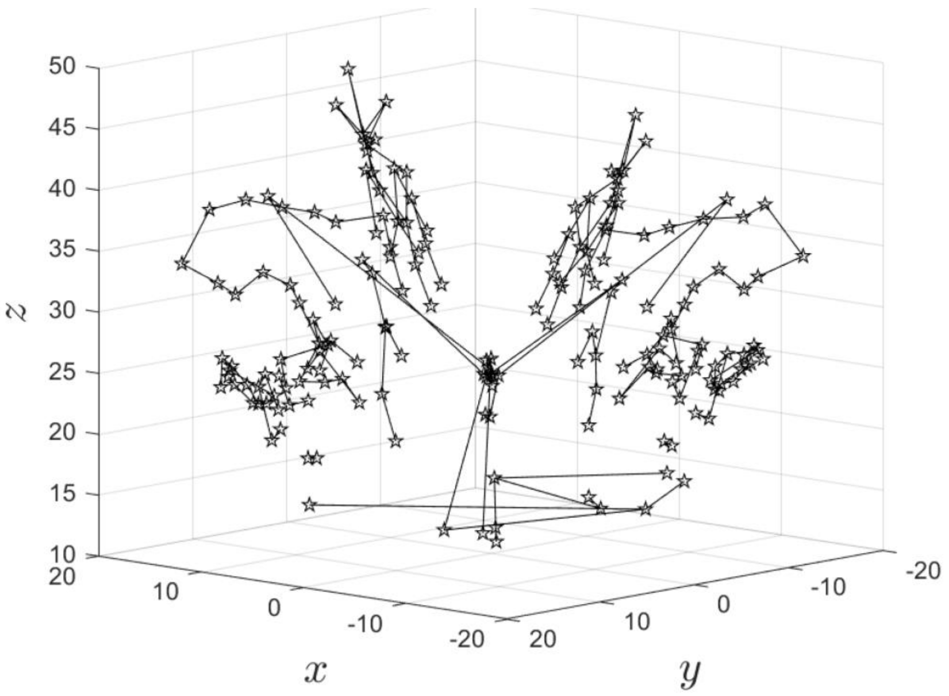
- creation
- **destruction**
- splitting
- merging
- merging → splitting



Random Templex

- How does it encode topological tipping points?

A better picture of how the holes are evolving in the system's phase space can be gained by using the coordinates of the barycenters of the holes for an immersion of D into this space. Each node is immersed in the phase space using the coordinates of the corresponding hole's barycenter.



A constellation C is the set of immersed nodes and edges forming a connected component in the digraph D of a random templex.

Constellations should lead to the equivalent of a stripex in a random templex, i.e. to non-equivalent paths that a nonlinear system follows when it is driven by multiplicative noise.

Random stripexes should provide us with the stretching, folding, squeezing and tearing mechanisms that knead the topological structure of a noise-driven flow.

Concluding remarks

There is no relationship between the algebraic structure of the equations governing a dynamic system and the type of flows they produce.

The mechanisms that shape a flow are topological in nature and are found in the full phase space.

***

Classical algebraic topology describes how a structure is built, but it says nothing about the flow associated with the dynamical system.

A templex is a new mathematical concept describing both structure and flow, as well as how these fundamental properties may change abruptly, i.e. the associated topological tipping points.

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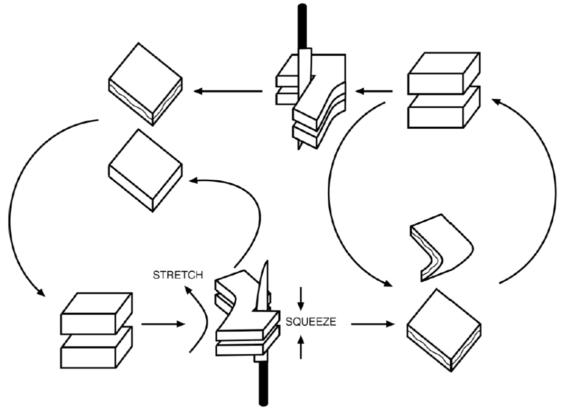
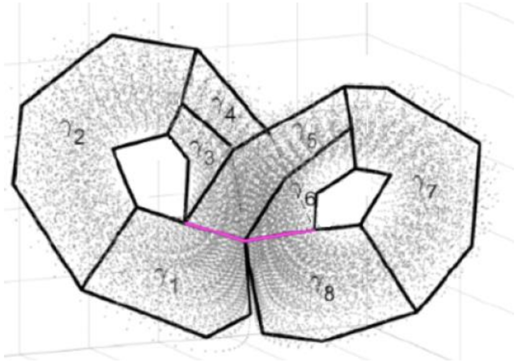
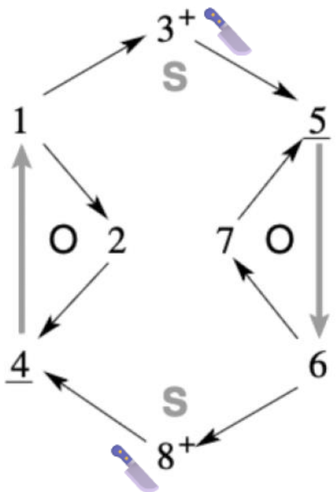
The templex approach provides the right way to look at dynamical systems, whether deterministic or stochastic.

The approach is currently under development in its theoretical, computational and applied contexts.







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